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TESIS DOCTORAL

ESSAYS ON TWO-SIDED MATCHING AND MECHANISM DESIGN

Autor:
Alfredo Salgado Torres

Director:
Dr. Antonio Romero Medina

Departamento:
Economía

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Firma del Tribunal Calificador:

Firma

Presidente:

Vocal:

Secretario:

Calificación:

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A mi familia.

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RESUMEN

En esta Disertación Doctoral se analizan problemas económicos en la intersección de las áreas de la Teoría de los Juegos, los Problemas de Emparejamiento y el Diseño de Mecanismos. En particular, he trabajado en el análisis de problemas de emparejamiento en esquemas descentralizados tales como Mercados de Trabajo, Problemas de Admisiones a la Universidad y Problemas Generales de emparejamiento. Recientemente, la investigación sobre estos problemas ha ganado un importante impulso, tanto teórica como empíricamente, debido al estudio de problemas cruciales que no ‘pueden ser analizados bajo los paradigmas clásicos de oferta y demanda. En mi disertación analizo dos problemas principales. Por un lado, hago un análisis de equilibrio de mecanismos descentralizados de emparejamiento con agentes que tomas sus decisiones estratégicamente. Por otro lado, también analizo el problema clásico de emparejamiento en presencia de externalidades, es decir, problemas de emparejamiento donde las preferencias de los agentes dependen de todos los emparejamientos factibles entre agentes y no solo del conjunto de potenciales parejas en un lado opuesto del mercado. Esta disertación se compone de los siguientes tres capítulos, que en sí mismos constituyen contribuciones independientes en esta área de la economía.

En el primer capítulo, titulado *Incomplete Information and Costly Signaling in College Admissions*, analizo el problema de admisión a las universidades con información incompleta acerca de las habilidades de los estudiantes. En este modelo considero, que universidades con calidades observables y estudiantes con información privada se emparejan de acuerdo con un mecanismo descentralizado donde los estudiantes pueden señalizar sus habilidades con señales que son costosas. Bajo estas condiciones, caracterizo un equilibrio simétrico separador del juego inducido por este mecanismo en el cual se maximiza el número de emparejamientos y los mejores estudiantes se matriculan en las mejores universidades. Esta caracterización del equilibrio permite realizar diversos ejercicios de estática comparativa, en ellos se muestra que el cambio de diversos parámetros del modelo afecta asimétricamente a los estudiantes. Por ejemplo, un incremento en el número de estudiantes lleva a que aquellos de baja cualificación reduzcan su inversión en señalización, mientras los de alta cualificación podrían aumentarla. Se observan patrones similares cuando los estudiantes enfrentan un incremento en el número de plazas escolares o un incremento en las calidades de las universidades. Finalmente, se analizan las ganancias del proceso de señalización comparando los pagos de equilibrio de este juego con uno en el que no hay señalización. Entre otros resultados, encuentro que bajo ciertas distribuciones de las habilidades de los estudiantes, una demanda suficientemente alta por las plazas de las universidades llevaría a todos los colegios a tener ganancias positivas.

En el segundo capítulo, titulado *Many-to-one Matching: externalities and Stability*, se analiza la existencia de asignaciones estables en problemas de emparejamiento con externalidades. En este problema, los agentes no solo toman en cuenta sus parejas

sino también las parejas de los demás agentes para determinar sus preferencias. Es decir, el emparejamiento de los demás afecta la valoración que yo tengo de mi propio emparejamiento. Una vez que estas externalidades se toman en cuenta, los agentes forman expectativas sobre el conjunto de emparejamientos que ellos consideran admisibles, dichas predicciones son llamadas *estimation functions*. Dado un conjunto de *estimation functions* φ, un emparejamiento es φ-stable si este es admisible para todos los agentes y no es bloqueado por ninguna coalición. Un primer resultado muestra que los emparejamientos φ-stables podrían no existir. Aun más, se muestra que ningún conjunto de *estimation functions* puede asegurar la existencia de emparejamientosφ-stable. Además se muestra que un emparejamiento φ-stable puede existir, en el caso en el que todos los emparejamientos sean considerados admisibles por todos los agentes, bajo una restrición de las preferencias de los agentes llamada *bottom q-substitutability*. Finalmente, se analiza también una noción del *core* en esta clase de problemas de emparejamiento con externalidades llamado el φ-core. Se demuestra que el φ-core y el conjunto de emparejamientos φ-stables siempre coinciden independientemente del conjunto de *estimation functions* φ.

Finalmente, en el tercer capítulo, titulado *A Simple Decentralized Matching Mechanism in Markets with Couples*, se analiza un mecanismo descentralizado de emparejamiento muy simple llamado *One Application Mechanism*. Bajo este mecanismo, se puede sostener en Equilibrio perfecto en Subjuego (SPE) cualquier emparejamiento estable del mercado. Sin embargo, se encuentra que también es posible sostener emparejamiento inestables en este tipo de equilibrio. Se muestra que solo un tipo muy especial de inestabillidad es admisible en SPE, se argumenta que esta inestabilidad tiene su origen en fallas de coordinación entre los miembros de una pareja. En mi principal resultado, se muestra que el *One Application Mechanism* implementa en SPE el conjunto de emparejamientos (pairwise) estables de mercados de emparejamiento con parejas...

ABSTRACT

This dissertation involves the analysis of economic problems in the intersection of the fields of Game theory, Two-sided Matching and Mechanism Design. In particular, I work on decentralized two-sided matching problems such as job markets, college admissions, marriage problems, etc. In recent years, these economic problems have become crucial, both theoretically and empirically, since there are many important real-world markets where the traditional supply-demand paradigm cannot be applied. My disertation includes the analysis of two main problems. On the one hand, I consider the equilibrium analysis of decentralized matching mechanisms with strategic decision makers. On the other hand, I also analyze two-sided matching problems with externalities where agents' preferences depend on the complete assignment between workers and firms, students and colleges, etc. and not only on the set of agents on the opposite side of the market. The dissertation consists of the following three chapters that constitute independent contributions in this area.

In the first chapter, entitled *Incomplete Information and Costly Signaling in College Admissions*, I analyze a problem of college admissions with incomplete information about student skills. Colleges with observable qualities and students with private information are matched according to a decentralized mechanism where students can signal their abilities with costly observable signals. I characterize a separating symmetric equilibrium of this game where the number of potential matches is maximized and the best students enroll at the best colleges. My closed form characterization allows to conduct meaningful comparative statics analyses. I show that the effect of a change in the underlying parameters of the model is not symmetric across students. For instance, an increase in the number of students leads the low skilled students to decrease the investment in signaling while the high skilled applicants may increase it. Similar patterns arise when students face a change in the number of school places or college qualities. Finally, I analyze the gains of the signaling process by comparing equilibrium payoffs between this separating equilibrium and a pooling equilibrium with no signaling. Among other results, I show that under certain distributions of the student skills, a large enough demand for school places leads all colleges to get positive gains.

In the second chapter, entitled *Many-to-one Matching: externalities and Stability*, I analyze the existence of stable assignments in matching problems with externalities. In this setting, agents not only care about whom they are matched with but also the partners of the others. Once externalities are considered, agents form expectations on the set of matchings that they consider admissible, such predictions are called *estimation functions*. Given a set *estimation functions* φ, a matching is φ-stable if it is admissible for every agent and not blocked by any coalition. I show that a φ-stable matching may not exist. Further, no set of *estimation functions* assures the existence of φ-stable matchings. In the case where all matchings are considered admissible for every agent, a φ-stable

matching exists under a restriction on agents' preferences called *bottom q-substitutability*. Finally, I analyze a notion of the core in matching problems with externalities called the φ-core. I find that the φ-core and the set of φ-stable matchings always coincides for any given set of *estimation functions* φ.

Finally, in the third chapter, entitled *A Simple Decentralized Matching Mechanism in Markets with Couples*, I analyze a simple decentralized matching mechanism called One Application Mechanism. Under this mechanism any stable matching of the market can be achieved in Subgame Perfect Equilibrium (SPE). However, I find that the mechanism may achieve also unstable matchings in SPE. I show that only one special kind of instability is admissible in equilibrium. Further, I argue that this instability only comes from coordination failures between members of couples. My main result shows that the One Application Mechanism implements in SPE the set of pairwise stable matchings of markets with couples. . .

1

INCOMPLETE INFORMATION AND COSTLY SIGNALING IN COLLEGE ADMISSIONS

1.1 INTRODUCTION

A decentralized college admissions process is associated with coordination problems. This not only means that some agents may end up unmatched, but also that the matching mechanism may be ineffective to assign the best students to the best colleges. Not only the grade coordination among agents could explain these failures of the matching process but also the presence of incomplete information. In real-world, college qualities seem to be observable for all agents, but students' abilities are rather private information. Thus colleges with observable qualities may be indifferent among many applicants depending on the available information about student abilities. These indifferences may lead colleges to accept applicants already accepted by other colleges and eventually remain unmatched at the end of the matching process.

The literature on this issue shows that some simple matching mechanisms can alleviate the effects of coordination problems in matching problems with complete information.[1] A crusial characteristic of these mechanisms is that they try to exhaust the possibility of matching agents in a stable fashion and under certain conditions maximize the number of potential matches of the problem. Some of these matching mechanisms assure the stability of equilibrium assignments by restricting students to send only one application.[2] However, when agents are allowed to send multiple applications, unstable assignments may end up in equilibrium. According to Triossi (2009), in this setting with multiple applications it is easy to restore the stability of equilibrium assignments by introducing a small application cost in the process. But even when the application cost is negligible, leading students to submit multiple applications, some dynamic mechanisms result effective to achive stable assignments in equilibrium.[3] Thus, in college admissions with complete information, it is relatively easy to guarantee the stability of equilibrium assignments and alleviate the problem of coordination.

In contrast, in incomplete information environments, we require additional conditions to alleviate the problem of coordination, since relevant characteristics of agents are not observable any more. In this setting, the role of signaling seems to be crucial to understand how colleges and students match each other in college admissions. For instance, Coles, et. al. (2010) introduce a cost-free signaling mechanism in decentralized matching problems with incomplete information about agents' preferences. Among other desirable properties, in equilibrium this mechanism increases the expected num-

1 These mechanisms are simple in the sense that they consist in only two stages. In the first stage, agents on one side of the market send (simultaneously) a proposal to the agents on the opposite side of the market. In the second stage, agents collect their offers and accept or reject (simultaneously) of of the available proposals.

2 See Alcalde, Perez-Castrillo and Romero-Medina (1998) and Alcalde and Romero-Medina (2000).

3 See Sotomayor, 2003; Romero-Medina and Triossi, 2010; and Haeringer, G. and Wooders, M., 2011.

ber of matches and the welfare of agents who signal their preferences. However, a cost-free signaling mechanism is not very realistic in decentralized college admissions. For instance, most selective colleges and universities in the US require a set of signals to measure students' abilities: the test scores of either the SAT or ACT,[4] essay questions, recommendation letters and personal interviews are the most important requirements. Hence, a costly signaling model seems to be a correct approach to analyze this problem, since a student has to spend significant amounts of effort (and money) in order have better signals and improve his chance of enrolling at college.

In this chapter, we analyze a matching problem where students want to enroll at colleges with observable qualities. Student abilities are private information, however all agent know the prior distribution of student skills. In this setting, students want to enroll at the best universities while colleges want to enroll high skilled students. Agents are matched according to a simple decentralized matching mechanism called **Costly Signaling Mechanism** (CSM) which runs in two stages. In the **signaling stage**, students choose a costly observable score to signal their abilities. In the **matching stage**, colleges and students are matched according to a simple two-stage matching process. First, colleges simultaneously make one offer to a student; and after that, students collect their offers and simultaneously choose one offer among the available ones. The CSM induces an extensive form game that is characterized by an equilibrium assignment and a signaling strategy.

In order to understand the effects of the presence of incomplete information in college admissions, we analyze the benchmark matching problem with no signaling. In this setting, all students are ex-ante identical, since colleges only know the prior distribution of student abilities. Under these conditions, we characterize a symmetric equilibrium of this game whose agents' payoffs depend on the number of students, the expected value of student skills and colleges' qualities. This equilibrium has several interesting implications. First of all, in equilibrium colleges expect to enroll average students, since the matching process does not provide any additional information about student abilities. Secondly, we find that the probability of enrolling a student is decreasing in college qualities. Then only the highest quality college fills its school seat with probability one while the rest of agents may end up unmatched with positive probability. Finally, we also show that an increase in the number of students increases colleges' payoffs while students' payoffs decrease.

Our main results regard with the analysis of a separating symmetric signaling equilibrium of the game induced by the CSM where all students play according to the same signaling strategy. To sustain this separating equilibria, we consider a set of beliefs by which higher scored students are associated with higher abilities. Under these beliefs, colleges form an interim ordinal preference relation on the set of students by which they prefer to enroll higher scored students. This implies that for each profile of student scores, there is a unique equilibrium assignment in the matching stage of the CSM that is consistent with these beliefs. This equilibrium assignment is assortative, i.e. the highest scored student is matched with the best college; the second highest scored student is matched with the second best college; and so on.

4 The SAT (Scholastic Assessment Test) and the ACT (American College Testing) are the most important standardized tests for college admissions in the USA.

In the signaling stage of the CSM, students take as given the assortative assignment of the matching stage and play a signaling game by choosing a costly observable score to signal their abilities. We characterize a symmetric pure strategy Nash equilibrium of this game. This equilibrium is characterized by a strictly increasing and continuous differentiable signaling strategy that depends on student skills. In equilibrium, no pair of students choose the same score, then this symmetric signaling equilibrium induces a unique equilibrium matching that is assortative with respect to the true student skills. So we find that in this separating equilibrium, the highest skilled student is matched with the best college; the second highest skilled student is matched with the second best college; and so on. Further, in this equilibrium it is maximized the number of potential matches and agents are induced to match efficiently in the sense that the best students enroll at the best colleges.

Our closed form characterization allows to conduct meaningful comparative static analyses. Our main result shows that the effect of a change in the underlying parameters of the model is not symmetric across students, such effect depends on student abilities. The first comparative statics exercise deals with the effect of a change in the number of students. Intuitively, an increase in the number of competitors (students) should decreases the probability of enrolling at college which leads students to decrease their investment in signaling. However, our results show that this effect is not symmetric, since a low skilled student has a decrease in this probability while a high skilled student may has an increase. Thus an increase in the number of students leads the low skilled students to decrease the investment in signaling while the high skilled students may increase it.

This result is useful to explain an interesting empirical fact observed in real-world college admissions. In the US is observed a decline in the mean SAT scores as the participation rate increases.[5] According to the College Board,[6] the mean SAT scores declines because more students of varied academic backgrounds are represented in the pool of test-takers. It is clear that this interpretation only considers the positive correlation between the SAT and students abilities to argue that an increase in the number of applicants systematically decreases the proportion of good test-takers in population. Our results suggest an alternative explanation based on the underlying signaling game of the problem. According to our model, an increase in the number of competitors not only leads low skilled students to decrease the investment in signaling but also increases the proportion of people who decide to do it. Then an increase in the number of competitors will eventually lead students to reduce the average investment in signaling with no change in the underlying distribution of student skills.

We also analyze the effect of a change in the number of school places and a change in college qualities. These two experiment have very similar implications, and as in the previous case, the effect of these experiments is not symmetric across students. In particular, we show that an increase in the number of school places (in college qualities) leads low skilled students to increase the investment in signaling while the high skilled students may decrease it.

5 California Postsecundary Education Commission (CPEC), "SAT Scores and Participation Rate" at http://www.cpec.ca.gov/StudentData/50StateSATScores.asp.

6 "43% of 2011 College-Bound Senior Met SAT College and Career Readness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

Finally, we analyze the gains of the CSM which are defined in a natural way as the difference in equilibrium payoffs between this separating signaling equilibrium of the CSM and a symmetric equilibrium of the problem with no signaling. We show that students' gains are strictly increasing with respect to student skills. However, this property of students gains does not guarantee to avoid potential losses. Further, it is possible to show that under certain distributions of skills all students can get negative gains.

Colleges' gains depend on the expected values of order statistics. Thus, the analysis of these gains is a difficult issue, since for most distributions there are no closed form solutions for moments of order statistics. We analyze the particular case of exponentially distributed skills that allows us to calculate a closed form solution for colleges' gains. Even when the exponential model is a very particular case, it has interesting implications. First of all, colleges' gains are monotonic increasing in college qualities, i.e. the best colleges have the greatest gains. Second, colleges' gains are monotonic increasing in the number of students, i.e. all colleges benefit from an increasing demand for school places. Finally, we show that a sufficiently large demand for school places leads all colleges to get positive gains.

The rest of the chapter is organized as follows; in Section 1.2, we describe the basic model and definitions; in Section 1.3, we analyze the benchmark college admissions problem with no signaling; in Section 1.4, we introduce the CSM and its equilibrium characterization; in Section 1.5, we conduct our exercises of comparative statics; in Section 1.6, we analyze the gains of the CSM; in Section 1.7, we present some conclusions of the chapter. Finally, all proofs of this chapter are in appendix A.

1.2 THE MODEL

There are $M \geq 1$ colleges and N students such that $M \leq N$. Let $C = \{c_1, c_2, ..., c_M\}$ denote the set of colleges with typical agent $c \in C$ and let $S = \{s_1, s_2, ..., s_N\}$ denote the set of students with typical agent $s \in S$. Each college $c \in C$ is characterized by an observable parameter $v_c > 0$, which is interpreted as the quality of the college c. With some abuse of notation, we use v_j to denote the quality of the college c_j. In order to simplify, we usually say "the student i" instead of "the student s_i" and "the college j" instead of "the college c_j". We assume without lost of generality (w.l.g) that colleges' qualities satisfy the following condition, $v_1 \geq v_2 \geq ... \geq v_M$.

Each student $s \in S$ is characterized by a parameter $\alpha_s \in [0, w]$ that denotes his skills or academic abilities. We say that a student s is more skilled than a student s' whenever $\alpha_s > \alpha_{s'}$. Students' skills are private information, this implies that only the student $s \in S$ knows the realization of his own parameter α_s. We assume that student skills are independently and identically distributed on some interval $[0, w]$ according to a strictly increasing and continuous differentiable cumulative distribution function F such that $F(0) = 0$ and $F(w) = 1$.[7] The distribution F has a continuous density $f = F'$ that satisfies $f(\alpha) > 0$ for all $\alpha \in (0, w)$. All elements of the model are common

7 All results hold when students' parameters are independently and identically distributed on the interval $[0, \infty)$ according to a strictly increasing and continuous differentiable cumulative distribution function F such that $F(0) = 0$ and $\lim F(w) = 1$ as $w \to \infty$.

knowledge, i.e. the distribution of skills F; the number of students and colleges; and colleges' qualities.

1.2.1 *The matching problem*

For simplicity, we focus on the simplest one-to-one matching problem,[8] i.e. each college has only one available school seat. In this setting, an assignment is a matching between colleges and students which is a mapping that specifies a partner for each agent, allowing the possibility that some agents remain unmatched. Formally

Definition 1 *A matching μ is a mapping from the set $S \cup C$ onto itself such that:*

1. *If $\mu(s) \neq s$ then $\mu(s) \in C$;*
2. *If $\mu(c) \neq c$ then $\mu(c) \in S$; and*
3. *$\mu(s) = c$ if and only if $s = \mu(c)$.*

According to this definition, a student (college) with no partner is matched with himself (itself). In order to simplify, each student (college) get an utility equal to the quality (skills) of the partner. Let $U_s(\mu)$ and $U_c(\mu)$ be the utilities of the student s and the college c, respectively, under the matching μ. Then each student $s \in S$ has a payoff,

$$U_s(\mu) = \begin{cases} v_c \text{ if } \mu(s) = c. \\ 0 \text{ Otherwise} \end{cases} \tag{1}$$

Each college $c \in C$ has a payoff,

$$U_c(\mu) = \begin{cases} \alpha_s \text{ if } \mu(c) = s. \\ 0 \text{ Otherwise} \end{cases} \tag{2}$$

We normalize the utility of the prospect of remaining unmatched to zero for both colleges and students.

In two-sided matching literature, a college admissions problem is described by a three-tuple $(S, C, \succ)$, where S is a set of students, C is a set of colleges and $\succ = (\succ_{s_1}, ..., \succ_{s_N}; \succ_{c_1}, ..., \succ_{c_M})$ denotes a profile of ordinal preferences. In this problem, each agent $a \in S \cup C$ has a complete, strict and transitive preference relation $\succ_a$ over the set of agents on the opposite side of the market and the prospect of remaining unmatched.

It is easy to see that each student $s \in S$ has a preference relation $\succ_s$ over the set of colleges and the prospect of remaining unmatched $C \cup \{s\}$, such that: a) $c \succ_s s$ if and only if $v_c > 0$ and b) for all $c, c' \in C$, it is satisfied that $c \succ_s c'$ if and only if $v_c > v_{c'}$. Since college qualities are observable, all students have identical ordinal preferences. In a similar way, each college $c \in C$ has a preference relation $\succ_c$ on the set of students and the prospect of having a position unfilled $S \cup \{c\}$, such that: a) $s \succ_c c$ if and only if $\alpha_s > 0$ and b) for all $s, s' \in S$, it is satisfied that $s \succ_c s'$ if and only if $\alpha_s > \alpha_{s'}$. Let $\succeq_a$ denote the weak preference relation associated with $\succ_a$ for each agent $a \in S \cup C$. Thus,

8 The model can be easily extended to many-to-one matching problems under the assumption that colleges form responsive preferences (Roth and Sotomayor, 1991).

for any $c, c' \in C$, $c \succeq_s c'$ implies either $c \succ_s c'$ or $v_c = v_{c'}$. In a similar way, for any $s, s' \in S$, $s \succeq_c s'$ implies either $s \succ_c s'$ or $\alpha_c = \alpha_{c'}$.

A matching μ is **individually rational** whenever $\mu(a) \succeq_a a$ for all $a \in S \cup C$. A student-college pair (s, c) such that $\mu(s) \neq c$ **blocks** the matching μ if, $s \succ_c \mu(c)$ and $c \succ_s \mu(s)$. A matching μ is **stable** if it is individually rational and not blocked by any student-college pair. Let $\mathcal{E}(S, C, \succ)$ denote the set of **stable matchings** of the college admissions problem $(S, C, \succ)$.

1.3 THE BENCHMARK PROBLEM: COLLEGE ADMISSIONS WITH NO SIGNALING

We analyze the benchmark problem of college admissions with no signaling and incomplete information about student skills. In this setting, all students are ex-ante identical, since colleges only know the prior distribution of student skills. So the expected value of student abilities $E[\alpha]$ is the best estimation of student skills.

We consider that colleges and students are matched according to the following simple decentralized matching mechanism in two stages.

1. **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, then the college c is making an offer to the student s. If $m(c) = c$, the college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers to the student s (note that a student always receives an offer from himself) ;

2. **Hiring:** Each student $s \in S$ chooses one of his available offers in $O(s)$.

Colleges and students play the game induced by this simple mechanism. In complete information environments with strict preferences, Alcalde and Romero-Medina (2000) show that this mechanism implements in **Subgame Perfect Equilibrium (SPE)** the set of stable matchings of college admissions problems. Thus this class of decentralized matching mechanisms exhausts the possibility of matching colleges and students in a stable way.[9] Further, under certain conditions on agents' preferences,[10] this mechanism also maximizes the number of potential matches.

In the presence of incomplete information, these results do not hold any more. The mechanism may has many equilibria depending on the available information about student abilities and the grade of coordination among colleges. In this section, we focus on two "natural" equilibria of the problem, with and without coordination among colleges, whose characterization permit us to analyze the effects of the presence of incomplete information and the problem of coordination in decentralized college admissions. The explicit characterization of agents' payoffs based on the number of students and colleges; college qualities and student skills allows us to identify the effect of a change in one of these parameters on equilibrium payoffs.

Before analyzing these equilibria, we argue that it is easy to characterize the equilibrium students' behavior. Since college qualities are observable, at any possible equilibria

9 These results hold in problems where colleges have quotas of students providing colleges' preferences are responsive, see Roth and Sotomayor (1991).

10 When colleges have responsive preferences respect to individual preferences, the set of agent unmatched and unfilled positions are the same at any stable matching (Roth and Sotomayor, 1990). This result implies that this simple matching mechanism not only exhausts the possibility of matching agents in a stable way, but also maximizes the number of potential matches when colleges have responsive preferences.

students must choose the best offer among the available ones. It is clear that under any alternative choice rule, students cannot get a better assignment. Then the rule where students choose the best offer among the available ones is a dominant strategy for students. We assume that colleges anticipate this optimal students' behavior and decide their offers. For simplicity, we label each student with one number from 1 to N. These labels are observable for all agents and do not provide information about student skills.

We analyze an equilibrium situation where colleges coordinate their actions based on students' labels. Consider a profile of strategies where students follow their dominant strategy while each college c_j sends one message to the student j. Let μ be the outcome matching of this strategy profile. It is easy to verify that this assignment satisfies $\mu\left(c_j\right) = j$ for $j = 1, ..., M$ while the rest of students remain unmatched, i.e. $\mu\left(j\right) = j$ for $j = M+1, ..., N$. Under this assignment, each college gets a payoff equal to $E\left[\alpha\right]$ while students get a payoff equal to v_j for $j = 1, ..., M$ and zero otherwise. It is easy to show that this profile of strategies is a SPE of this college admissions game. First of all, note that no student has a profitable deviation, since students are following their dominant strategy. Secondly, a college c_k can deviate by sending a message to any alternative student $j \neq k$. In this case, the college c_k either will get matched with another student $j = k+1, ..., N$ or will be rejected by another student $j = 1, ..., k-1$. It is clear that this deviation cannot be profitable, since all students are ex-ante identical. Further, note that under this equilibrium, it is maximized the number of potential matches. This equilibrium is well defined for any permutation of the set of students and further these equilibria are payoff equivalent for colleges.

Now we consider an equilibrium of this game with no coordination among colleges. We want to show that the profile of strategies where students choose the best offer among the available ones and colleges make one offer to each student with equal probability is a SPE of this college admissions game.

We consider a college admissions problem with $M \geq 1$ colleges and $N \geq M$ students. As before, we label each student with one number from 1 to N with no additional information about student skills but the prior distribution. Assume that each college sends one offer to each student with equal probability (i.e. $\frac{1}{N}$), we want to show that no college has a profitable deviation from this strategy. Consider that any college c_j with observable quality v_j is planning to deviate from this strategy. Note that there are $j-1$ colleges with higher quality and $M-j$ colleges with lower quality than the college c_j. Since college qualities are observable, an offer of this college c_j always beats any other offer of the $M-j$ lower quality colleges. Then one offer of the college c_j will be accepted by a student i whenever every of the $j-1$ higher quality colleges make one offer to any of the other $N-1$ students.

Thus the total number of combinations of offers from $M-j$ colleges to N students is N^{M-j} while total number of combinations of offers from $j-1$ colleges to $N-1$ students is $\left(N-1\right)^{j-1}$. Since colleges do not coordinate, one offer of the college c_j will be accepted by the students i with probability:

$$\frac{\left(N-1\right)^{j-1} N^{M-j}}{N^{M-1}} = \left(\frac{N-1}{N}\right)^{j-1} \quad \text{for } j = 1, ..., M. \tag{3}$$

Then by making an offer to the student i with probability $\frac{1}{N}$, the college c_j will get an expected payoff $\left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right]$. Note that this payoffs are independent of student skills, since all students are ex-ante identical.

Now consider that the college c_j is planning to deviate from this strategy by making one offer to each student i with probability $p_i \neq \frac{1}{N}$. It is easy to see that such deviations cannot be profitable, since $\sum_{i=1}^{N} p_i^* \left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right] = \sum_{i=1}^{N} p_i \left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right]$ for any $p_i \neq \frac{1}{N}$ such that $\sum_{i=1}^{N} p_i = 1$. Then the profile of strategies where colleges send one offer to each student with equal probability is a symmetric SPE of this game. Note that in this equilibrium, colleges' payoffs EQ_{c_j} depend on the number of students, the expected value of student skills and the rank of colleges.

$$EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right] \text{ for } j = 1, ..., M. \tag{4}$$

Now we deduce students' payoffs in this symmetric equilibrium. In this case, we have to find the probability that each student $i = 1, ..., N$ enrolls at college c_j for $j = 1, ..., M$. First of all, we know that the student i will reject any available offer but the best one. This implies that a student i enrolls at the college c_1 with probability $\frac{1}{N}$. It is easy to show that , in general, a student i enrolls at the college c_j with probability, $\frac{1}{N}\left(\frac{N-1}{N}\right)^{j-1}$. Then the expected payoff of the student $i = 1, ..., N$ is given by,

$$EU\left(N, M\right) = \frac{1}{N}\sum_{k=1}^{M} v_k \left(\frac{N-1}{N}\right)^{k-1} \tag{5}$$

Since students enroll at each college with positive probability, the student payoffs are strictly positive for any $M \geq 1$ and $N \geq M$ and satisfies $v_1 > EU\left(N, M\right) > 0$. In addition, students may remain unmatched with positive probability equal to $1 - \frac{1}{N}\sum_{k=1}^{M}\left(\frac{N-1}{N}\right)^{k-1} = \left(\frac{N-1}{N}\right)^{M}$, which is strictly positive, increasing in the number of students and decreasing in the number of school places.

This simple model is useful to analyze the main consequences of the absence of coordination in college admissions with incomplete information. First of all, note that for any number of students N and school places M, all agents remain unmatched with positive probability but the highest quality college. Note that the college c_1 fills its vacancy with probability one and gets a expected payoff equal to $E\left[\alpha\right]$ which is the best prediction of student skills without additional information. Second, the equilibrium assignment may be inefficient, since colleges only know the expected value of student skills. Further, the probability of enrolling a students is decreasing in the rank of colleges, since the probability $\left(\frac{N-1}{N}\right)^{j-1}$ is strictly decreasing in j. Therefore, the absence of coordination mainly damages low quality colleges.

1.4 THE COSTLY SIGNALING MECHANISM.

In this section, we analyze a decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under this mechanism, each student $s \in S$ chooses a costly observable score $P_s \geq 0$ to signal his skills. Hence, a student $s \in S$ with type α who chooses a score P_s has to pay the cost

$$C(\alpha, P_s) = \frac{c(P_s)}{\phi(\alpha)} \tag{6}$$

We assume that the function $c(\cdot)$ is strictly increasing, continuous differentiable and convex such that $c(0) = 0$. We also consider that the function $\phi(\cdot)$ is strictly increasing, continuous differentiable and bounded in the interval $[0, w]$ such that $\phi(0) > 0$.

The profile of student scores $(P_s)_{s\in S}$ is observable for all agents. Under the CSM, colleges and students are match according to the following decentralized matching procedure in two stages:

1. **Signaling Stage:** Each student $s \in S$ with parameter α chooses a score $P_s \geq 0$ at the cost $C(\alpha, P_s)$.

2. **Matching Stage:** After observing the profile of scores $(P_s)_{s\in S}$, students and colleges match according to the following decentralized matching process:

   a) **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, the college c is making an offer to the student s. If $m(c) = c$, the college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers of the student s (a student always receives an offer from himself) ;

   b) **Hiring:** Each student $s \in S$ chooses one of his available offers $O(s)$.

We want to characterized a symmetric and strictly separating equilibrium where all students use the same signaling strategy. Obviously, the model can admit many other symmetric equilibria. For instance, pooling equilibria where no student invests in signaling (in this situation we could sustain any of the symmetric equilibria analyzed in the previous section) or semi-separating equilibria.

To sustain the strictly separating equilibrium, we consider beliefs according to which the higher scored students are associated with higher academic skills. Formally, we describe these beliefs by a continuous distribution of student abilities given the score $P > 0$, i.e. a continuous cumulative distribution $G(\alpha \mid P)$. We assume that these beliefs have associated a continuous density $g(\alpha \mid P)$ and satisfy $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$. Note that these conditions imply that $E[\alpha \mid P'] > E[\alpha \mid P]$ for all $\alpha \in (0, w)$ whenever $P' > P$ where $\int \alpha g(\alpha \mid P)\, d\alpha = E[\alpha \mid P]$. Thus higher scored students are associated with higher expected skills.

The payoffs of colleges are given by the expected quality of enrollees that depends on the outcome of the CSM (i.e. a matching between colleges and students) and the profile of student scores. So given a matching μ, a college $c \in C$ has expected payoffs $E\left[\alpha \mid P_{\mu(c)}\right]$. On the other hand, a student $s \in S$ with parameter α gets payoffs $v_c - C(\alpha, P_s)$ such that $\mu(s) = c$ and $-C(\alpha, P_s)$ otherwise.

In order to simplify, we consider that after observing the profile of student scores $(P_s)_{s\in S}$, colleges "update" their ordinal preferences in the following simple way. Each college $c \in C$ forms an auxiliary preference relation $\succ_c^*$ over the set students and the prospect of remaining unmatched, $S \cup \{c\}$ such that: a) $s \succ_c^* c$ if and only if $P_s > 0$ and b) for any $s, s' \in S$, $s \succ_c^* s'$ if and only if $P_s > P_{s'}$. Note that the profile of interim preferences$\succ_C^* = (\succ_c^*)_{c\in C}$ is consistent with beliefs $G(\alpha \mid P)$.

At the matching stage of the CSM, the profile of student scores is given, so colleges form the interim preference $\succ_C^* = (\succ_c^*)_{c\in C}$ while the the signaling cost is sunk for students. This implies that student preferences at this stage are well defined and coincide with the strict preferences $\succ_S = (\succ_s)_{s\in S}$. Assume w.l.g. that colleges have different qualities, i.e $v_1 > v_2 > .. > v_M > 0$. On the other hand, assume also that students' scores satisfy $P_s \neq P_{s'}$ for all $s, s' \in S$. This assumption about student scores is not strong, since in equilibrium ties will happen with probability zero. Then for any given profile of student scores $(P_s)_{s\in S}$, at the matching stage of the CSM there is a well defined college admissions problem with strict preferences denoted by $(S, C, (\succ_S, \succ_C^*))$. Under these conditions, we can establish the following result.

Proposition 1 *Consider the beliefs $G(\alpha \mid P)$ such that such that $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$. Then for any profile of student scores $(P_s)_{s\in S}$ such that $P_s \neq P_{s'}$ for all $s, s' \in S$ and $s \neq s'$, there is a unique SPE outcome in the second stage of the CSM. This equilibrium outcome is the unique stable matching of the college admissions problem, $(S, C, (\succ_S, \succ_C^*))$. Further, this unique equilibrium assignment is assortative.*

Only stable matchings between students and colleges are a reasonable outcomes of the CSM (Alcalde and Romero-Medina, 1998). Further, under the interim college preferences $\succ_C^* = (\succ_c^*)_{c\in C}$ the outcome matching is assortative, i.e. the highest scored student is matched with the best college; the second highest scored student is matched with the second best college, and so on.

In the following section, we analyze the signaling equilibrium of the first stage of the CSM. We focus on a symmetric pure strategy equilibrium where all students play according to the same signaling strategy. We analyze a settings with $M \geq 1$ school places and $N > M$ students. However, the model can be easily extended to analyze any problem with the same number of students and school places.

1.4.1 *The signaling equilibrium*

We analyze the signaling equilibrium of the first stage of the CSM. To characterize this equilibrium, we take as given the outcome of the matching stage of the mechanism. To illustrate the problem, we focus on the simplest case with only one college with quality $v_1 > 0$ and $N > 1$ students.

In this setting, we analyze a separating symmetric Nash equilibrium of the signaling game played by students. This equilibrium is characterized by a continuous differentiable and strictly increasing signaling strategy with respect to student abilities. We focus on the student 1's problem, who chooses a score P_1 to signal his skills while the rest of students play according to a signaling strategy $\rho : [0, w] \to \Re_+$ which is assumed to be strictly increasing and continuous differentiable in α such that $\rho(0) = 0$.

Since the outcome matching of the CSM is assortative, the students 1 with parameter α gets the payoff $v_1 - C(\alpha, P_1)$ whenever $P_1 > \rho(\alpha_i)$ for all $i \neq 1$. This happens with probability $\Pr[P_1 > \rho(\alpha_2), ..., P_1 > \rho(\alpha_N)] = F\left(\rho^{-1}(P_1)\right)^{N-1}$ given that student skills are identically and independently distributed according to F. Otherwise, the students 1 gets the payoff $-C(\alpha, P_1)$ with probability $1 - F\left(\rho^{-1}(P_1)\right)^{N-1}$. Hence, the expected payoffs of the student 1 with parameter α, when the rest of students play according to the signal function $\rho(\cdot)$ is:

$$\pi(\alpha, P_1) = v_1 F\left(\rho^{-1}(P_1)\right)^{N-1} - C(\alpha, P_1) \tag{7}$$

The student 1 takes as given the signaling strategy of the rest of students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. The first order condition (FOC) with respect to P_1 leads the following condition,

$$v_1 (N-1) F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right) \frac{1}{\rho'(\rho^{-1}(P_1))} - \frac{c'(P_1)}{\phi(\alpha)} = 0 \quad (8)$$

By reordering the previous expression, we obtain the following differential equation,

$$v_1 (N-1) \phi(\alpha) F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right) = c'(P_1) \rho'\left(\rho^{-1}(P_1)\right) \quad (9)$$

In a symmetric equilibrium, $P_1 = \rho(\alpha)$ then the previous differential equation becomes in the following,

$$v_1 (N-1) \phi(\alpha) F(\alpha)^{N-2} f(\alpha) = c'(\rho(\alpha)) \rho'(\alpha) \quad (10)$$

By solving this differential equation with the initial condition $\rho(0) = 0$, we find that the equilibrium signaling strategy satisfies,

$$\rho(\alpha) = c^{-1}\left(v_1 (N-1) \int_0^{\alpha} \phi(x) F(x)^{N-2} f(x)\, dx\right) \quad (11)$$

The equilibrium signaling strategy $\rho(\cdot)$ is strictly increasing and continuous differentiable in α. Note that $\rho(\cdot)$ only satisfies the FOC of the student 1's maximization problem, which is necessary but not sufficient to characterize the signaling equilibrium. Hence, we have to prove that the signaling strategy $\rho(\cdot)$ is in fact a symmetric equilibrium of this game. The equilibrium payoff of any student with parameter α is given by,

$$\pi(\alpha, \rho(\alpha)) = v_1 F(\alpha)^{N-1} - \frac{c(\rho(\alpha))}{\phi(\alpha)} \quad (12)$$

It is not difficult to show that this payoff function satisfies $\frac{d}{d\alpha}(\pi(\alpha, \rho(\alpha))) > 0$ and $\pi(0, \rho(0)) = 0$. Hence, we show that any alternative score $P' \neq \rho(\alpha)$ cannot be a profitable deviation for any student with parameter α. Consider that a student with parameter α is planning to choose another score $0 < P' < \rho(\alpha)$ while the rest of students are playing according to the signaling strategy $\rho(\alpha)$. Since the signaling strategy is strictly increasing in α and satisfies $\rho(0) = 0$, there exists a unique $0 < \alpha' < \alpha$ such that $\rho(\alpha') = P'$. This implies that a student who chooses an alternative strategy $P' = \rho(\alpha')$ will get a payoff, $\pi(\alpha, P') = \pi(\alpha, \rho(\alpha'))$. Hence, a student with parameter α losses the extra payoff $\pi(\alpha, \rho(\alpha)) - \pi(\alpha, \rho(\alpha'))$ by not deviating to $\rho(\alpha') = P'$. Then

$$\pi(\alpha, \rho(\alpha)) - \pi\left(\alpha, \rho(\alpha')\right) = v_1 \left(F(\alpha)^{N-1} - F\left(\alpha'\right)^{N-1}\right) - \frac{c(\rho(\alpha)) - c(\rho(\alpha'))}{\phi(\alpha)} \quad (13)$$

It is not difficult to show that the extra payoffs $\pi(\alpha, \rho(\alpha)) - \pi(\alpha, \rho(\alpha'))$ can be reduced to the following expression,

$$v_1\left(F(\alpha)^{N-1}-F(\alpha')^{N-1}\right)-\frac{1}{\phi(\alpha)}v_1(N-1)\int_{\alpha'}^{\alpha}\phi(x)F(x)^{N-2}f(x)\,dx \tag{14}$$

Since the function $\phi(x)$ is positive, strictly increasing in x and bounded in $[0,w]$, it is clear that the following inequality holds,

$$\frac{1}{\phi(\alpha)}v_1(N-1)\int_{\alpha'}^{\alpha}\phi(x)F(x)^{N-2}f(x)\,dx < v_1\int_{\alpha'}^{\alpha}(N-1)F(x)^{N-2}f(x)\,dx \tag{15}$$

In addition note that by definition $\int_{\alpha'}^{\alpha}(N-1)F(x)^{N-2}f(x)\,dx = F(\alpha)^{N-1}-F(\alpha')^{N-1}$. This last condition implies that $\pi(\alpha,\rho(\alpha))-\pi(\alpha,\rho(\alpha'))>0$ for all $\alpha'<\alpha$ which proves that $P'=\rho(\alpha')$ is not a profitable deviation. By a similar argument, it is possible to show that any alternative score $P''>\rho(\alpha)$ cannot be a profitable deviation. Then the signaling strategy $\rho(\alpha)$ is a symmetric equilibria of the signaling game played by students. In the following section, we show that all of these results hold in the general case with $M\geq 2$ colleges and $N>M$ students. All proofs and calculations can be found in the Appendix.

1.4.1.1 *The general case: $N > M \geq 2$.*

Now consider a general case with N students and M colleges such that $N>M\geq 2$. Assume w.l.g. that all colleges have different qualities and satisfy $v_1>v_2>...>v_M>0$. As before, we analyze the student 1's maximization problem with parameter $\alpha\in(0,w)$ while all other students play according to some signaling strategy $\rho_M:[0,\infty)\to\Re_+$. As before, we assume that the signaling strategy $\rho_M(\cdot)$ is strictly increasing and continuous differentiable in α such that $\rho_M(0)=0$.

The student 1 chooses a score $P_1\geq 0$ to signal his abilities. Consider the following notation, we say that the student 1 has a "success" whenever $P_1>\rho_M(\alpha_i)$ for some other student $i\neq 1$ and a "failure" whenever $P_1<\rho_M(\alpha_i)$ for some other student $i\neq 1$. The probability of having one "success" is $F\left(\rho_M^{-1}(P_1)\right)$ whereas the probability of having one "failure" is $1-F\left(\rho_M^{-1}(P_1)\right)$. Note that these probabilities are independent, since students' parameters are independently distributed.

For any given number of students $N\geq M$, the student 1 with score P_1 enrolls at the colleges c_j with quality v_j, whenever he has $N-j$ "successes" and $j-1$ "failures". Note that this situation happens $\binom{N-1}{j-1}$ different times, then the probability of enrolling at the college c_k is,

$$\binom{N-1}{k-1}F\left(\rho_M^{-1}(P_1)\right)^{N-k}\left[1-F\left(\rho_M^{-1}(P_1)\right)\right]^{k-1}. \tag{16}$$

The previous argument implies that the probability of enrolling at the college $c_j\in C$ follows a binomial distribution. Hence, the expected payoff of the student 1 $\pi(\alpha,P_1)$ satisfies,

$$\pi(\alpha,P_1)=\sum_{k=1}^{M}v_k\binom{N-1}{k-1}F\left(\rho_M^{-1}(P_1)\right)^{N-k}\left[1-F\left(\rho_M^{-1}(P_1)\right)\right]^{k-1}-C(\alpha,P_1) \tag{17}$$

The student 1 takes as given the signaling strategy of the rest of students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. In Appendix A, we solve the student 1's maximization problem in a symmetric equilibrium where all students play according to the same signaling strategy $\rho_M(\alpha)$. We show that the signal function that satisfies the FOC of the student 1's maximization problem characterizes this symmetric separating equilibrium. Thus we establish the following result,

Proposition 2 *The signaling strategy,*

$$\rho_M(\alpha) = c^{-1}\left(\sum_{k=1}^{M-1}(v_k - v_{k+1})\int_0^{\alpha}\phi(x)\, f_{(k,N-1)}(x)\, dx + v_M\int_0^{\alpha}\phi(x)\, f_{(M,N-1)}(x)\, dx\right)$$

is a symmetric equilibrium of college admissions problems with $M \geq 2$ colleges and $N > M$ students.

Proof. See appendix A. ■

Given the equilibrium signaling strategy $\rho_M(\cdot)$, a student with parameter α will get expected payoffs,

$$\pi(\alpha, \rho_M(\alpha)) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F(\alpha)^{N-k}\left[1 - F(\alpha)\right]^{k-1} - \frac{c(\rho_M(\alpha))}{\phi(\alpha)} \tag{18}$$

Note that to characterize the signaling equilibrium, we assume some desirable properties of the equilibrium signaling strategy. We should show that these properties are satisfied in equilibrium. A simple observation is enough to show that the equilibrium signaling strategy and agents' payoff are continuous differentiable functions in α. In the following result, we establish some interesting properties of these signaling strategy and equilibrium payoffs.

Proposition 3 *The equilibrium signaling strategy $\rho_M(\alpha)$ and student payoffs $\pi(\alpha, \rho_M(\alpha))$ satisfy the following properties:*

1. *$\rho_M(\alpha)$ is strictly increasing in α and bounded above.*
2. *$\pi(\alpha, \rho_M(\alpha))$ is strictly increasing in α.*

Proof. See appendix A. ■

Since the equilibrium signaling strategy is strictly increasing and probability of having two students with the same skills is zero, no pair of students will choose the same score. Then the equilibrium outcome of the CSM will be assortative with respect to the true student skills. Hence, the highest skilled student will be matched with the best college; the second highest skilled student will be matched with the second best college; and so on. Further, students with higher abilities get greater payoffs. This result comes from the single crossing property of the signaling cost, since higher skill students have lower marginal signaling cost.

On the other hand, the assortative structure of the equilibrium assignment of the CSM implies that colleges payoffs depend on the ranking of enrollees and the prior distribution of student skills. Let μ^* be the unique equilibrium outcome of the **CSM**, then in equilibrium colleges get expected payoffs,

$$EQ^*_{c_j} = E\left[\alpha \mid P_{\mu^*(c)}\right] = E\left[\alpha_{(j,N)}\right] = \int_0^w \alpha f_{(j,N)}(\alpha)\, d\alpha \text{ for } j = 1, ..., M. \tag{19}$$

Where $\alpha_{(j,N)}$ is the j-th order statistic from a sample of size N such that $\alpha_{(1)} = \max_{1 \leq i \leq N} \alpha_i$, $\alpha_{(2)}$ = second greatest α_i, and so on. It is well known that the order statistic $\alpha_{(j,N)}$ is distributed according to the probability density function,

$$f_{(j,N)}(\alpha) = \frac{N!}{(j-1)!\,(N-j)!} f(\alpha)\, F(\alpha)^{N-j} \left[1 - F(\alpha)\right]^{j-1} \text{ for } j = 1, ..., M. \tag{20}$$

Under this conditions, it is not difficult to show that responding to students' signals is an equilibrium situation for colleges. First of all, it is not difficult to show that the best strategy for any college c_j is to respond to students' signals providing all higher quality colleges $\{c_1, c_2, ..., c_{j-1}\}$ do. The argument is very simple, college c_j has to compare the expected skills of enrollees between responding to students' signals and any alternative admission rule. Note that c_j knows that all students are willing to accept its offer but those already enrolled at colleges $\{c_1, c_2, ..., c_{j-1}\}$, since by assumption those colleges respond to signals and have greater qualities. This implies that any potential enrollee of the college c_j has skills $\alpha \leq \alpha_{(j)}$. By responding to students' signals, the college c_j will enroll the best student among the available ones. In contrast, with any other admission rule it will enroll a lower skilled students.

Now consider the case of the best college, c_1 knows that its offer will be accepted by any student. Since by responding to students' signals, c_1 will enroll the best student among all students, it optimally responds to students' signals. Then a simple induction argument shows that all colleges respond to students' signals.

1.5 COMPARATIVE STATICS.

In previous sections, we characterize a separating symmetric equilibrium of the signaling game induced by the CSM. This equilibrium is characterized by an equilibrium signaling strategy that depends on several parameters like the prior distribution of skills; the number of students and school seats; and college qualities. Our explicit characterization allows us to conduct interesting comparative statics exercises to analyze the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. In particular, we focus on three experiments:

1. A change in the number of students;
2. A change in the number of school places; and
3. A change in the quality of colleges.

The analysis of these experiments is crucial to understand real-world colleges admissions as a signaling process whose outcome depends on the interaction of strategic decision makers. Our model provides a good approach to analyze the effect of a change of those underlying parameters.

One of the most important real-world signal in college admissions is the SAT test in the US. Most students take the SAT during the last year of high school, and almost all colleges and universities use it to make admission decisions. Empirical studies analyze the importance of the SAT and provide empirical evidence that support our model of decentralized college admissions with incomplete information and costly signaling. First of all, it is possible to argue that the SAT is a costly signal that depends on on the amount of effort invested by students. Further, according to the structure of the SAT is expected that higher skilled students get better scores. Secondly, it is well known that there is a strong correlation between SAT scores and real student skills. For instance, Frey and Detterman (2004) show that there is a high correlation between SAT scores and several measures of student success like IQs. Finally, the matching between colleges and students tend to be assortative with respect to the true student skills, since the best colleges and universities tend to enroll students with high SAT scores (Webster, 2001a, 2001b).

It is clear that an incorrect understanding of the underlying signaling game in college admissions may lead us to suggest wrong policy recommendations. For instance, empirical evidence in the US shows that there is a decline in the mean SAT scores as the participation rate increases. If we only consider the high correlation between SAT scores and measures of student skills. We can suggest that the decline in SAT scores comes from an increase in the proportion of low skilled students who have taken the test, i.e. a change in the current distribution of student skills. According to this argument, a policy recommendation would be to increase the expenditure in SAT coaching and tutorials in high school in order to improve the abilities of test-takers. However, the previous argument and policy recommendation ignore the underlying signaling game in college admissions, since they do not consider that in the face of new competitors, students may decrease strategically the investment in signaling with no change in the underlying distribution of skills.

1.5.1 *A change in the number of students.*

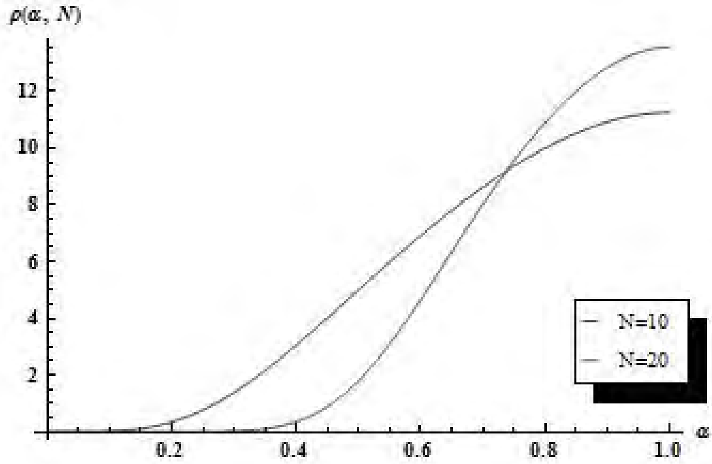
We analyze the effect of a change in the number of students over the equilibrium signaling strategy $\rho_M(\alpha) = \rho_M(\alpha, N)$. Intuitively, an increase in the number of students should decreases the probability of enrolling at college which leads students to decrease their investment in signaling. This intuition seems correct, however we are not considering that the effect of increasing the competition may not be symmetric across students. The probability of beating potential competitors to enroll at college depends on student abilities. A low skilled student knows that the probability of facing new high skilled competitors is big while a high skilled student knows that the probability of facing qualified competitors is small. We consider an exercise where we increase the number of students while we maintain fixed all other parameters of the model. The result of this exercise shows that this effect depends on academic abilities. Formally,

Proposition 4 *For any college admissions problem with $M \geq 1$ colleges and $N > M$ students, $\rho_M(\alpha, N+1) < \rho_M(\alpha, N)$ for all $\alpha \leq \alpha_N(N)$ and $N > M$. Further, the threshold $\alpha_N(N)$ is strictly monotone increasing N.*

Proof. See appendix A. ■

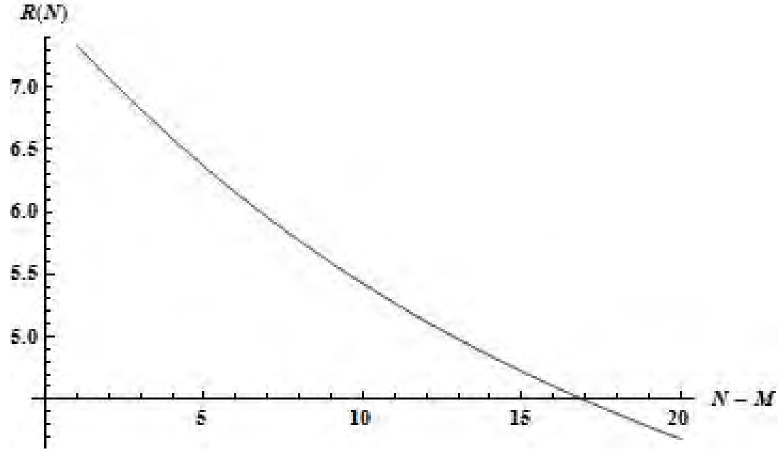
The previous result has two main implications. First of all, we find that as the number of students increases the low qualified students decrease the investment in signaling while the high skilled students may increase it. A student with parameter α enrolls at college c_k with probability $\binom{N-1}{k-1} F(\alpha)^{N-k}\left[1-F(\alpha)\right]^{k-1}$. Thus we can identify two opposite effects. On the one hand, the probability $F(\alpha)^{N-k}\left[1-F(\alpha)\right]^{k-1}$ of having $N-k$ successes decreases as N increases. On the other hand, the number of successful draws $\binom{N-1}{k-1}$ where the student beats $N-k$ competitors increases as N increases. Then these two opposite effects may lead the high (low) skilled students to increase (decrease) the probability of enrolling at college and increase (decrease) the investment in signaling.

Figure 1: Effect of increasing the number of students



The second interesting implication deals with the monotonicity of the threshold $\alpha_N(N)$. We find that this threshold is monotone increasing in N, i.e. $\alpha_N(N+1) > \alpha_N(N)$ for any $N > M$. This implies that students do not increase the investment in signaling when they face $N+1$ competitors if they have already decreased it with N. This property of the equilibrium and the fact that the the equilibrium signaling strategy is bounded above allow us to infer the effect on the average investment in signaling when the number of students increases. Let $R(N) = \int \rho_M(\alpha, N) f(\alpha)\, d\alpha$ be the expected (average) investment in signaling, according to the previous argument there should exist a sufficiently large demand for school places $\hat{N}$ such that $R(N+1) < R(N)$ for all $N \geq \hat{N}$.

Figure 2: Average investment in signaling with respect to N



In the US college admissions system, it has been extensively analyzed the impact of increasing the number of test-takers on the mean SAT scores. According to data of the College Board for several years, it has been observed a decline in the mean SAT scores as the participation rate increases. The College Board explains this stylized fact in the following way[11]:

"*It is common for mean scores to decline slightly when the number of students taking an exam increases because more students of varied academic backgrounds are represented in the test-taking pool*."

This interpretation only considers the positive correlation between the SAT and students abilities to infer that an increase in the number of applicants systematically decreases the proportion of good test-takers. However, this interpretation does not consider the underlying signaling game in college admissions. In contrast, our model suggests that an increase in the number of applicants not only leads the low skilled students to decrease the investment in signaling but also increases the proportion people who reduce it. Then an increase in the number of competitors will eventually reduce the average investment in signaling with no change on the prior distribution of student abilities.

1.5.2 *A change in the number of school places.*

The following experiment deals with the effect of a change in the number of school places on the equilibrium signaling strategy. We consider an experiment where the number of school places can increase but remaining lower than the number of students. Intuitively, an increase in the number of available school places should increase the probability of enrolling at college and lead students to increase the investment in signaling.

Our model shows that this intuitive argument may not be correct, at least not for all students. As in the previous case, the effect of a change in the number of school places is not symmetric across student. In order to simplify, we consider a very simple model

11 College board (2011), "43% of 2011 College-Bound Senior Met SAT College and Career Readiness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

with N students and one college that offers $M \geq 1$ school seats. In this setting, it is easy to show that the equilibrium signaling strategy of the problem satisfies the following,

$$\rho_M(\alpha, M) = c^{-1}\left(v_1 \int_0^{\alpha} \phi(x) f_{(M,N-1)}(x)\, dx \right) \tag{21}$$

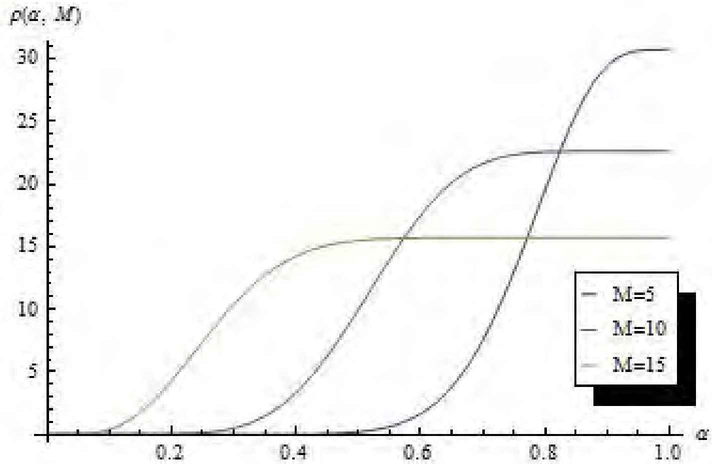
Then we establish the following result,

Proposition 5 *For any college admissions problem with one college with $M \geq 2$ school seats and $N > M+1$ students, $\rho_M(\alpha, M+1) > \rho_M(\alpha, M)$ for all $\alpha \leq \alpha_M(M, N)$. Further, the threshold $\alpha_M(M, N)$ is monotone increasing in N and monotone decreasing in M.*

Proof. See appendix A. ■

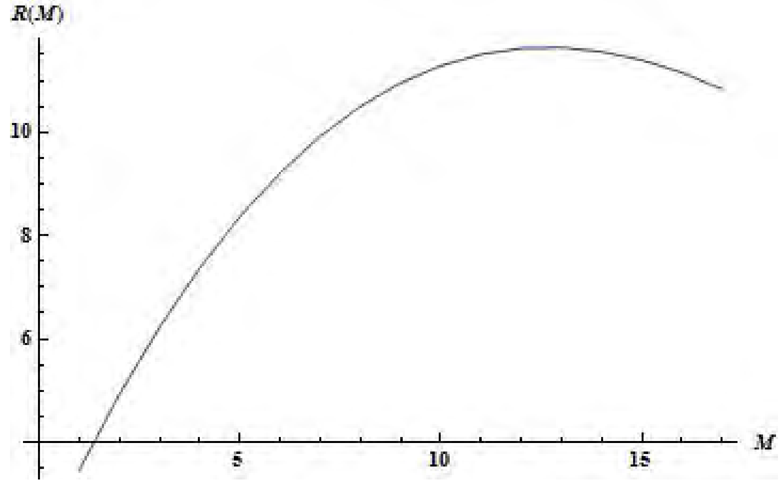
The previous result has several interesting implications. First of all, an increase in the number of school places leads the low skilled students to increase the investment in signaling while the high qualified students may decrease it. Intuitively, an increase in the number of school places should be equivalent to a decrease in the number of students with a fixed number of school seats.

Figure 3: Effect of increasing the number of school places



We also show that threshold $\alpha_M(M, N)$ is monotone decreasing in M, i.e. $\alpha_M(k+1, N) < \alpha_M(k, N)$ for $k = 1, ..., M$. This result allows us to establish some general conclusions about the shape of the average investment in signaling as a function of the number of school places. We prove that low and high skilled students change the investment in signaling in opposite directions. Then when there are a few available school seats, a new one is very valuable and leads students to increase the average investment in signaling. However, when the number of school places increases, the proportion of people that decrease the investment also increases, since a new school place is less valuable. Thus, there should be a big enough number of school places, from which an additional school seat decreases the average investment in signaling.

Figure 4: Average investment in signaling with respect to M



1.5.3 *A change in the quality of colleges*

In this section, we analyze the effect of a change in college qualities on the equilibrium signaling strategy. We focus on a change in qualities that preserve ordinal student preferences. For instance, if the college c_k changes its quality from v_k to v'_k, it should be true that $v_{k-1} > v'_k > v_{k+1}$, whenever $v_{k-1} > v_k > v_{k+1}$. This assumption makes comparable the equilibrium signaling strategies before and after the change in college qualities, since the equilibrium assignment of the CSM is the same in both situations.

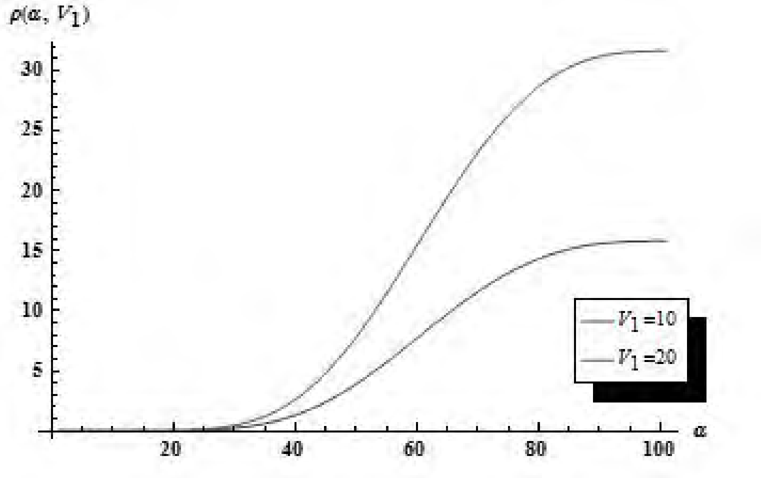
Intuitively, when a college increases its quality the average quality of schools also increases, this increment in students' valuations makes reasonable to increase the investment in signaling. However, as in the previous cases, this result depends on student abilities. Let $sgn\,(x)$ be a function such that $sgn\,(x) = 1$ if $x > 0$, $sgn\,(x) = -1$ if $x < 0$ and $sgn\,(x) = 0$ if $x = 0$. Then we establish the following result.

Proposition 6 *For any college admissions problem with $M \geq 2$ colleges and $N > M$ students, $sgn\left(\rho_M\left(\alpha, v'_k\right) - \rho_M\left(\alpha, v_k\right)\right) = sgn\left(v'_k - v_k\right)$ for all $\alpha \leq \alpha_{v_k}(N,k)$ and $k = 1, ..., M$. Further, the threshold $\alpha_{v_k}(N,k)$ is monotone increasing N for all $k = 2, ..., M$ and monotone decreasing in k.*

Proof. See appendix A. ■

The previous result has interesting implications. First of all, only low skilled students are willing to increase the investment in signaling while the high qualified students may decrease it. Intuitively, the an increase in college qualities is more valuable for low skilled students than for high skilled ones. This implies that only an increase in the quality of the best colleges leads the highest skilled students to increase the investment in signaling. On the other hand, we also show that the threshold $\alpha_{v_k}(N,k)$ is monotone decreasing in k, i.e. $\alpha_{v_k}(N,k+1) < \alpha_{v_k}(N,k)$ for all $k = 1, ..., M-1$. As expected, students are more willing to increase their investment for high quality colleges. Further, in appendix A we show that $\alpha_{v_1}(N,1) = w$ for any N, which implies that only an increase in the quality of the college c_1 has no asymmetry across students, i.e. all students are willing to increase the investment in signaling.

Figure 5: A change in the quality of the best college



1.6 GAINS OF THE CSM

In previous sections, we analyze a separating symmetric equilibria of the CSM that maximizes the number of potential matches and lead the best students to enroll at the best colleges. In contrast with no signaling, the low quality colleges are able to enroll better students with positive probability. Then some colleges may prefer to run an admissions system with no signaling to enroll better students with positive probability. A similar argument applies for low skilled students, whom pay the signaling cost and lose the chance of enrolling at high quality colleges.

According to the previous argument, some agents may get losses under the CSM in the sense that they can get better assignments and payoff with no signaling. Further, it seems that low quality colleges and low skilled students are the most damaged agents under the CSM. We define the gains of implementing the CSM in a natural way, as the difference in equilibrium payoffs between the separating signaling equilibria of the CSM and the symmetric equilibria of college admissions problem with no signaling. According to this definition, students' gains are defined as follows,

$$L(\alpha) = \pi(\alpha, \rho_M(\alpha)) - EU(N, M) \tag{22}$$

Since the student's payoff in the game with no signaling $EU(N,M)$ is type independent, students' gains are strictly increasing in α. However, there always exits a proportion of people that gets losses under the CSM, since $\pi(0, \rho_M(0)) = 0$ and $EU(N,M) > 0$. Note that eventually all students may get losses depending on the prior distribution of skills. However, only the highest skilled students have the possibility of getting positive gains.

The previous definition implies that college c_j's gains are defined as follows,

$$\Delta EQ(j, N) = EQ^*_{c_j} - EQ_{c_j} \text{ for } j = 1, ..., M. \tag{23}$$

Where $EQ^*_{c_j} = E\left[\alpha_{(j)}\right]$ is c_j's payoff in the separating equilibria of the CSM and $EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E[\alpha]$ is c_j's payoff in the game with no signaling. The analysis of these gains is not a trivial issue. For instance, it is clear that with no signaling an increase in the number of applicants must increase the probability of enrolling a student with average abilities. Under the CSM, an increase in the number of applicants should lead colleges to enroll better and better students. Since colleges gains are the difference between these payoffs, it is not clear the final effect of increasing the number of students on colleges' gains. Similar arguments can be applied in the case of qualities, it is not clear which colleges get the highest gains or if colleges gains are monotone in college rankings.

On the other hand, to analyze colleges' gains we require the analysis of order statistics. This is a difficult problem, since most distributions have no closed form solutions for moments of order statistics. To analyze this problem, we consider particular prior distributions where it is possible to find closed form formulas for these moments. We focus on the exponential model[12], in this case it is possible to show that $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1, ..., M$ where $E[\alpha] = \theta$ (Huang, 1974). Then, colleges' gains $\Delta EQ(j, N)$ satisfy the following equation,

$$\Delta EQ(j, N) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N}\right)^{j-1} \tag{24}$$

Then, we establish the following result.

Proposition 7 *Consider any M by N college admissions problem such that $N > M \geq 1$. Assume that students' skills are exponentially distributed with parameter $\theta > 0$. Then the following holds:*

1. *$\Delta EQ(j, N)$ is strictly monotone increasing in N, i.e. $\Delta EQ(j, N+1) > \Delta EQ(j, N)$ for all $N > M$ and $j = 1, ..., M$;*

2. *$\Delta EQ(j, N)$ is strictly monotone decreasing in j, i.e. $\Delta EQ(j+1, N) < \Delta EQ(j, N)$ for all $N > M$ and $j = 1, ..., M-1$; and*

3. *For any $M \geq 1$ there always exists an $N^* > M$ such that $\Delta EQ(j, N) \geq 0$ for all $j = 1, ..., M$ and all $N \geq N^*$.*

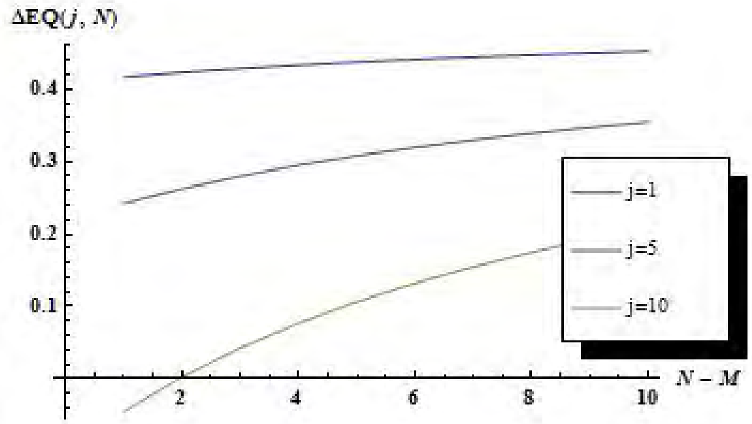
Proof. See appendix A. ■

The previous result has interesting implications. First of all, an increasing demand for school places improve the gains of enrolling students based on costly signals. Intuitively, an increasing pool of students leads colleges to reduce the risk of remaining unmatched while a costly signal becomes more and more effective to pick the best available students. Another interesting implication regards with the comparison of gains among colleges. As in the case of students, colleges' gains can be ranked according to college qualities. This result implies that the big winners of the CSM are the high quality colleges, which not only enroll the best students but also get the greatest gains. The third interesting implication regard with the relationship between the size of the

12 Skills are exponentially distributed with parameter $\theta > 0$, if α is distributed according the density function, $f(\alpha; \theta) = \frac{1}{\theta} e^{-\frac{\alpha}{\theta}}$. In this case, the cumulative distribution function is $F(\alpha; \theta) = 1 - e^{-\frac{\alpha}{\theta}}$. In addition, $E[\alpha] = \theta$.

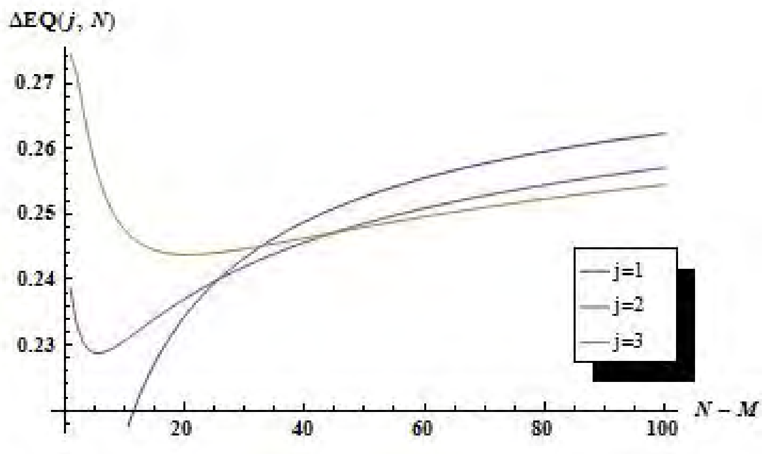
demand for school places and colleges' gains. We find that a big enough demand for school places leads all colleges to get positive gains. This result contrasts with the case of students, where there is a proportion of students that always get losses.

Figure 6: Colleges' gains with exponential distributed skills



It is easy to show that the previous results cannot be trivially extended to any prior distribution of student skills. As we show in the following figure, we cannot guarantee neither the monotonicity of colleges' gains with respect to the number of students nor the monotonicity with respect to colleges' qualities. In this case, we consider Beta distributed skills with parameters $a = 10$ and $b = 2$. Note that this distribution is skewed to the right, this fact may explain why the previous results about colleges' gains do not hold any more, since the probability of enrolling a good student with no signaling is significantly big.

Figure 7: Colleges' gains with Beta (10,2) distributed skills



1.7 CONCLUSION

We analyze some consequences of coordination problems in decentralized college admissions with incomplete information. We consider a matching problem where colleges with observable qualities want to enroll student whose abilities are private information. We analyze a simple decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under the CSM, students choose a costly observable score to signal their skills. We characterize a separating symmetric equilibrium of the game induced by the CSM. In this equilibrium the CSM maximizes the number of potential matches of the problem and induces agents to matched efficiently, in the sense that the best students will enroll at the best colleges. Hence, for the case in which the number of students equals the number of school seats, all agents will get matched while when there are more students than school places only the highest skilled students will get matched.

We conduct three exercises of comparative statics that allow us to analyze the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. Our main result shows that this effect is not symmetric across students, since they depend on student abilities. The first comparative statics exercise regards with the effect of a change in the number of students. In this case, we show that an increase in the number of students leads low skilled students to decrease the investment in signaling while the high skilled students may increase it. We also analyze the effect of a change in the number of school places and a change in college qualities with similar implications.

Finally, we analyze the gains of the CSM which are defined in a natural way as the difference in equilibrium payoffs between the separating signaling equilibria of the CSM and the symmetric equilibria of the college admissions problem with no signaling. Under this definition, students' gains are strictly increasing with respect to the student skills, but eventually, all students may get losses depending on the prior distribution of skills. Since colleges' gains require the analysis of order statistics, we consider the particular case of exponential distributed skills which allows us to find closed form formulas of colleges' gains. The exponential model has very interesting implications. First of all, colleges' gains are monotone increasing in college qualities. Second, colleges' gains are monotone increasing in the number of students, i.e. all colleges benefit from an increasing demand for school places. Finally, we show that a sufficiently large demand for school places leads all colleges to get positive gains.

2 MANY-TO-ONE MATCHING: EXTERNALITIES AND STABILITY

2.1 INTRODUCTION

In standard two-sided matching problems such as job markets, college admissions and marriage problems, agents on one side of the market have preferences over the set of agents on the opposite side. This specification of agents' preferences entails the absence of externalities, since agents only care about whom they are matched with. The absence of externalities simplifies the matching problem and lead us to establish relevant results regarding the existence and properties of several solution concepts in matching markets such as the set of stable matchings and the core.

In many applications agents not only care about whom they are matched with but also the partners of the others, so the presence of externalities may be crucial to understand real-world matching problems. We can find several real-world examples of matching markets that entail the presence of externalities. For instance, several competitive economic situations like tournaments and contests; markets with downstream competition; industrial research and development, etc.

The analysis of matching markets with externalities is challenging for at least two reasons. First of all, agents should consider the whole matching between firms and workers in order to define their preferences. Thus, in the presence of externalities agents' preferences must be defined over the set of all feasible matchings instead of the agents on the opposite side of the markets. This issue has crucial implications to analyze the matching problem, since some crucial restrictions over the domain of preferences may not be well defined in the presence of externalities.

The second issue is the solution concept. In standard matching problems, agents who plan to block a matching must compare their current and posterior partners in order to evaluate whether deviating is profitable or not. Under this notion of stability, deviating agents do not care about the reaction of non-deviating agents. In contrast, once externalities are considered a deviation may be profitable or not depending on the expected reaction of the rest of agents. This implies that agents must anticipate the reactions of non-deviating agents whenever they plan to block a current matching. According to this argument, it is clear that different assumptions about agents' predictions may lead to different notions of stability.

In previous literature, matching markets with externalities have been analyzed under two main approaches. In the first case, we try to establish some assumptions regarding agents' reactions that assure the existence of stable matchings without any restriction over the domain of agents' preferences. Under the second approach, we try to establish some restrictions over agents' preferences that guarantee the existence of stable matching under specific assumptions about agents' reactions. In general, both approaches are not equivalent.

Sasaki and Toda (1996) were the first to analyze the marriage problem with externalities. They propose a notion of stability that is similar to the idea of a conjectural equilibrium.[1] Under this solution concept, agents predict the set of matchings that they consider admissible under a given conjecture about the reactive behavior of agents. These predictions are called "*estimation functions*". Thus for a given set of *estimation functions* ', a matching is '-stable whenever it is admissible for every agent and not blocked by any man-woman pair or individual agent. Sasaki and Toda (1996) show that '-stable matching may not exists under particular sets of estimation functions. However, they claim that a '-stable matching exists if and only if all feasible matchings are considered admissible by every agent. In this case, we say that the set of *estimation functions* satisfy a condition called *full admissibility*. Hafalir (2008) extends this model by providing a set of endogenous *estimations* that depends on agents' preferences. Hafalir's estimations not only guarantees the existence of '-stable matchings but also shows that the assumption of *full admissibility* is not necessary to assure the existence of '-stable matchings.

Mumcu and Saglam (2010) also analyze the one-to-one matching problem with externalities. In this case, they propose a notion of stability that satisfies the following two *conditions*: a) deviating pairs join together while their previous mates, if any, divorce and b) the rest of agents remain matched as before the deviation. Mumcu and Saglam (2010) show that under this notion of stability a stable matchings may not exist. However, they propose restrictions on agents' preferences that assure the existence stable assignments. Bando (2010) provides similar results in many-to-one problems with externalities only on the firms' side.

This chapter deals with the analysis of many-to-one matching markets with externalities. In particular, we analyze the existence of stable matchings in the sense of Sasaki and Toda (1996). As in previous literature, we find that a φ-stable matching may not exist. Further, we show that no set of *estimation functions* (endogenous or exogenous) assures the existence of φ-stable matchings in many-to-one problems. This impossibility result contrasts with the case of marriage problems, where there is at least one set of estimation functions that guarantees the existence of φ-stable matchings.[2] According to these results, we can analyze our problem in two different ways to guarantee the existence of φ-stable matchings. On the one hand, we can fix a set of *estimations functions* to find reasonable restrictions on the domain of preferences. On the other hand, we can fix a restriction on the domain of preferences to find a set of *estimations functions*.

According to the previous argument, we consider a benchmark model where the set of *estimations functions* satisfies the condition of *full admissibility*. Under this restriction on agents' estimations, we find that a φ-stable matching exists whenever firms' preferences satisfy a restriction called *bottom q-substitutability*. This restriction generalizes the condition of q-substitutability (Cantala, 2004) to matching problems with externalities.

We also analyze whether the condition of *full admissibility* is necessary under the domain of *bottom q-substitutable* preferences. We consider a model with *pessimistic* agents to rationalize the set of estimation functions.[3] In this setting, we provide a set of endogenous *pessimistic estimation functions* that depends on agents' preferences. We show that this estimations functions do not satisfy the condition of full admissibility.

1 A conjectural equilibrium is a situation where no agent has incentives to deviate given a conjecture about the reactive behavior of agents (Rubinstein and Wolinsky, 1994; Azrieli, 2009).

2 See, Sasaki and Toda (1996) and Hafalir (2009).

3 Sasaki and Toda (1996) show that a φ-stable matching may not exist when only one of the agents is not pessimistic enough.

Further, we show that under the set of *pessimistic estimations*, a φ-stable matching exists providing preferences are *bottom q-substitutab*le.

The last part of the chapter deals with the analysis of the core of matching markets with externalities. Sasaki and Toda (1996) introduce a notion of the core in marriage problems with externalities. They show that the core and the set of pair-wise φ-stable matchings do not coincide. Further, the core may be empty for some instances of the problem. We propose an alternative notion of the core that depends on the set of estimation functions called the φ-core. Our main result shows that for any set of *estimation functions* the set of φ-stable matchings and the φ-core always coincide. This result contrasts with previous findings and implies that all properties of the set of stable φ-matchings naturally extend to the φ-core.

The rest of the chapter is organized as follows. In Section 2.2, we introduce the model and some basic examples; in Section 2.3, we introduce the condition of *bottom q-substitutability* to characterize the existence of φ-stable matchings; in Section 2.4, we introduce the set of *pessimistic estimations functions*; in Section 2.5, we introduce the φ-core; in Section 2.6, we present some conclusions. Finally, all proofs are in appendix B.

2.2 THE MODEL

Let F denote the set of firms and let W denote the set of workers. F and W are disjoint and finite sets with $m \geq 1$ and $n \geq 1$ members, respectively. Each worker $w \in W$ wants to work for at most one firm while each firm $f \in F$ has a quota $q_f \leq n$ that denotes the maximum number of workers that the firm is able to hire. We denote by $H_f = \left\{S \in 2^W : |S| \leq q_f\right\}$ the set of all subsets of workers (including the empty set of workers) that the firm f is able to hire. A matching is a rule that specifies a group of workers for each firm and a firm for each worker allowing for the possibility that some agents remain unmatched. Formally,

Definition 2 *A matching is a mapping* $\mu : F \cup W \to 2^{F \cup W}$ *such that:*

1. $|\mu(w)| = 1$ *for all* $w \in W$ *and either* $\mu(w) \cap F \neq \varnothing$ *or* $\mu(w) = \{w\}$*;*
2. $\mu(f) \in H_f$ *for all* $f \in F$*. If* $\mu(f) = \varnothing$ *then the firm* f *does not hire any worker; and*
3. $\mu(w) = \{f\}$ *if and only if* $w \in \mu(f)$*.*

Let $\mathcal{M}$ denote the set of all feasible matchings given F, W and q. In standard matching problems, agents have ordinal preferences over the set of agents on the opposite side of the market, i.e. workers have preferences over the set of firms and the prospect of remaining unmatched while firms have preferences over the set of groups of workers including the empty set. This specification of preferences lead agents to only care about whom they are matched with and not the partners of the others, i.e. it is not considered the presence of potential externalities. In this paper, we consider a more general setting where agents' valuations over potential partners may depend on the whole matching between firms and workers. According to this model, agents not only care about whom they are matched with but also the partners of the others. This specification of agents' preferences introduces explicitly the presence of externalities in matching problems.

In the presence of externalities, agents preferences would be defined by a preference relation over the set of all feasible assignments. Formally, each agent $a \in F \cup W$ has a complete, strict and transitive preference relation over the set of all feasible matchings $\mathcal{M}$ denoted by P_a^*. Thus for any given two pair of feasible matchings $\mu, \mu' \in \mathcal{M}$, the preference relation $\mu P_a^* \mu'$ means that the agent $a \in F \cup W$ prefers the assignment $\mu(a)$ under the matching μ to the assignment $\mu'(a)$ under the matching μ'. Note that this preferences are even more general than a simply comparison between agents (sets of agents) on the opposite side of the market, since an agent $a \in F \cup W$ may have the same assignment under two different matchings μ and μ' (i.e. $\mu(a) = \mu'(a)$) without being indifferent between them. According to this preferences, an agent would be indifferent between two matchings only if these matchings are identical. Formally, two matchings $\mu, \mu' \in \mathcal{M}$ are identical $\mu = \mu'$ if and only if $\mu(a) = \mu'(a)$ for all $a \in F \cup W$. For each agent $a \in F \cup W$, let R_a^* denote the weak preference relation induced by P_a^*, so for any two feasible matchings $\mu, \mu' \in \mathcal{M}$, the preference relation $\mu R_a^* \mu'$ means either $\mu P_a^* \mu'$ or $\mu = \mu'$. Let $P^* = \left(P_{f_1}^*, ..., P_{f_m}^*; P_{w_1}^*, ..., P_{w_n}^*\right)$ denote the profile of agents' preferences, thus a matching problem with externalities is a four-tuple (F, W, P^*, q).

We consider a solution concept for matching markets with externalities based on the concept of estimation functions (Sasaki and Toda, 1996; Hafalir, 2008). According to this solution concept, in the presence of externalities agents form expectations about the set of matching that they consider admissible. Agents consider such expectations in order to evaluate whether deviating from a current matching is profitable or not. These expectations are called *estimation functions* (or simply *estimation*s).

Before introducing a formal definition of the set of estimation functions, we require some additional notation. Let $A(f, S) = \{\mu \in \mathcal{M} : \mu(f) \in H_f\}$ denote the set of all feasible matchings where the firm f and the feasible set of workers S are matched. In a similar way, let $A(w, a)$ denote the set of all matchings where the worker $w \in W$ and the agent $a \in F \cup \{w\}$ are matched. For each firm $f \in F$ and any set of feasible workers $S \in H_f$, a *estimation function* φ_f specifies a non-empty subset of matchings where f and S are matched. Formally, for each $f \in F$, φ_f maps a non-empty subset of matchings $\varphi_f(S) \subset A(f, S)$ for every $S \in H_f$. In a similar way, for each worker $w \in W$ the *estimation function* φ_w maps a non-empty subset of matchings $\varphi_w(a) \subset A(w, a)$ for every $a \in F \cup w$. Let $' = \{(\varphi_f(\cdot), \varphi_w(\cdot)) : f \in F \text{ and } w \in W\}$ denote a set of *estimation functions* of a matching problem with externalities (F, W, P^*, q). Note that the set of *estimation functions* may be either exogenous given or an endogenous mapping which depend on agents preferences and other additional conditions.

As we argue before, our notion of stability in the presence of externalities depends on the concept of estimation functions. On the one hand, a first requirement for a stable matching is to be admissible for all agents. Intuitively, a stable matching has to be consistent with the fact that agents only care about matchings that they consider admissible. Hence, we introduce the following definition,

Definition 3 *Given a set of estimation functions ', a matching μ is φ-admissible whenever $\mu \in \varphi_a(\mu(a))$ for all $a \in F \cup W$.*

A second requirement for a stable matching is to be free of the presence of coalitions of agents that would like to deviate from this matching. In order to be consistent with the presence of externalities, agents must consider their estimation fictions when they plan to deviate from a current matching. Intuitively, an agent is willing to block a matching only if he is better off under all admissible matchings after deviating. According to this intuitive argument, we consider the following notion of deviating coalitions in the presence of externalities.

Definition 4 *An individual worker $w \in W$, such that $\mu(w) \neq w$, blocks the matching μ if $\mu' P^*_w \mu$ for all $\mu' \in \varphi_w(w)$.*

Definition 5 *A coalition firm-set of workers $\{f, S\}$ such that $S \in H_f$ and $\mu(f) \neq S$, blocks the matching μ if:*
1. *$\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(S)$ and*
2. *$\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(f)$ and all $w \in S$.*

Then given a set of *estimations functions* ′, a matching μ is φ-stable if it is φ-admissible and not blocked by any individual worker or coalition. Let $\mathcal{E}_\varphi(F, W, P^*, q)$ denote the set of φ-stable matchings of the problem.

Note that this notion of stability with externalities induces a natural notion of stability in matching problem without externalities where the concept of admissibility is not necessary.

2.2.1 *Preliminary examples*

In this section, we analyze some simple examples to introduce two main issues. First of all, we show that a φ-stable matching may not exist in matching problems with externalities. Secondly, we also provide a crucial result that establishes that no set of *estimation functions* guarantees the existence of φ-stable matchings.

Along these examples, we consider a particular set of *estimation functions* where every matching is considered admissible for all agents, this situation is called *full admissibility*. Formally, we say that a set *estimation functions* ′ satisfies *full admissibility* if for each firm $f \in F$, it is satisfied $\varphi_f(S) = A(f, S)$ for all $S \in H_f$ and for each worker $w \in W$, it is satisfied $\varphi_w(a) = A(w, a)$ for all $a \in F \cup \{w\}$. Let $\mathcal{E}(F, W, P^*, q)$ denote the set of φ-stable matchings with a set of estimation functions that satisfy *full admissibility*.

According to our notion of stability, in matching problems with externalities every agent who plans to deviate from a current matching should consider the set of all admissible matchings after deviating. In contrast, when there are no externalities, agents only consider their current and posterior partners to evaluate whether deviating is profitable or not. Thus, it seems to be more difficult to block a matching in the presence of externalities, since every deviating agent should be better off under all admissible matchings after deviating. This intuitive argument implies that it should be relatively easy to sustain the existence of stable matching in problems with externalities. However, it is easy to find examples of matching problems with externalities with no φ-stable matchings.

Example 1 *Consider a matching problem with three workers $W = \{w_1, w_2, w_3\}$ and two firms $F = \{f_1, f_2\}$ with quotas $q_{f_1} = 2$ and $q_{f_2} = 1$. In order to simplify, we describe a matching by a list of workers' partners in the order $w_1, w_2, ..., w_n$. For instance, the the matching $\mu_1 = f_1, f_1, f_2$ means that $\mu_1(w_1) = f_1$, $\mu_1(w_2) = f_2$ and $\mu_1(w_3) = f_2$. The set of all feasible matchings of this problem is given in the following table,*

Table 1: Set of feasible matchings

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, w_2, f_2$	$\mu_3 = w_1, f_1, f_2$	$\mu_4 = w_1, w_2, f_2$
$\mu_5 = f_1, f_1, w_3$	$\mu_6 = f_1, w_2, w_3$	$\mu_7 = w_1, f_1, w_3$	$\mu_8 = f_1, f_2, f_1$
$\mu_9 = w_1, f_2, f_1$	$\mu_{10} = f_1, f_2, w_3$	$\mu_{11} = w_1, f_2, w_3$	$\mu_{12} = f_1, w_2, f_1$
$\mu_{13} = w_1, w_2, f_1$	$\mu_{14} = f_2, f_1, f_1$	$\mu_{15} = f_2, f_1, w_3$	$\mu_{16} = f_2, w_2, f_1$
$\mu_{17} = f_2, w_2, w_3$	$\mu_{18} = w_1, f_1, f_1$	$\mu_{19} = w_1, w_2, w_3$	

Consider the following agents' preferences over the set of feasible matchings.

$$P^*_{f_1} = \mu_1, \mu_5, \mu_4, \mu_{11}, \mu_{17}, \mu_{19}, \mu_{10}, \mu_2, \mu_6, \mu_7, \mu_3, \mu_{15}, \mu_{13}, \mu_{16}, \mu_9, \mu_{14}, \mu_{18}, \mu_8, \mu_{12}.$$

$$P^*_{f_2} = \mu_2, \mu_3, \mu_4, \mu_1, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{17}, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_5, \mu_6, \mu_7, \mu_{12}, \mu_{13}, \mu_{18}, \mu_{19}.$$

$$P^*_{w_1} = \mu_1, \mu_2, \mu_5, \mu_6, \mu_8, \mu_{10}, \mu_{12}, \mu_{17}, \mu_{16}, \mu_{15}, \mu_{14}, \mu_3, \mu_4, \mu_7, \mu_9, \mu_{11}, \mu_{13}, \mu_{18}, \mu_{19}.$$

$$P^*_{w_2} = \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{18}, \mu_{15}, \mu_{14}, \mu_7, \mu_5, \mu_3, \mu_1, \mu_{19}, \mu_{17}, \mu_{16}, \mu_{13}, \mu_{12}, \mu_6, \mu_4, \mu_2.$$

$$P^*_{w_3} = \mu_8, \mu_9, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{16}, \mu_{18}, \mu_5, \mu_6, \mu_7, \mu_{10}, \mu_{11}, \mu_{15}, \mu_{19}, \mu_{17}, \mu_1, \mu_2, \mu_3, \mu_4.$$

Consider a set of estimation functions that satisfies full admissibility. Hence, every matching is φ-admissible and, as consequence, a candidate to be φ-stable. In order to show that a matching μ is not φ-stable, we need to find either a worker or a coalition with incentives to block the matching μ. Consider for instance the case of the matching $\mu_8 = f_1, f_2, f_1$ and the coalition composed by the firm f_1 and the empty set of workers. It is easy to show that the set of matchings where the firm f_1 and the empty set of workers are matched is $A(f_1, \emptyset) = \{\mu_4, \mu_{11}, \mu_{17}, \mu_{19}\}$. Note that by assumption, every matching in the set $A(f_1, \emptyset)$ is admissible for the firm f_1. Further, according to f_1's preferences it is easy to observe that $\mu' P^*_{f_1} \mu_8$ for all $\mu' \in A(f_1, \emptyset)$. Hence, the matching μ_8 cannot be φ-stable, since it is blocked by the coalition $\{f_1, \emptyset\}$. By a similar argument, it is easy to show that according to agents' preferences every feasible matching of this problem can be blocked by at least one coalition or worker.

Table 2: Blocking coalitions

Matching	Blocked by
$\mu_1, \mu_2, \mu_3, \mu_4$	$\{w_3\}$
$\mu_5, \mu_6, \mu_7, \mu_{12}$	$\{f_2, \{w_2\}\}$
$\mu_8, \mu_9, \mu_{10}, \mu_{14}, \mu_{15}$	$\{f_1, \emptyset\}$
μ_{11}, μ_{18}	$\{f_2, \{w_1\}\}$
$\mu_{13}, \mu_{16}, \mu_{17}, \mu_{19}$	$\{f_1, \{w_1, w_2\}\}$

This simple example shows that a φ-stable matching may not exist even when the set of estimation functions satisfy full admissibility. This is an interesting implication, since in contrast with the marriage problem with externalities the condition of *full admissibility* is not sufficient to assure the existence of φ-stable assignments in many-to-one matching problems (Sasaki and Toda, 1996).

It is interesting to note that our notion of stability also allows to analyze an some particular cases of the matching problem where we only consider the presence of externalities over one side of the market. For instance, we can consider a simple setting with externalities on the firms' side, i.e. firms have preferences defined over the set of all feasible matchings while workers have preferences over the set of firms and the prospect of remaining unmatched.

Example 2 *Consider a matching problem with two firms* $F = \{f_1, f_2\}$ *and three workers* $W = \{w_1, w_2, w_3\}$ *with quotas* $q_{f_i} = 2$ *for* $i = 1, 2$. *The set of all feasible matchings of this problem is given in the following table,*

Table 3: Set of feasible matchings

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, f_2, f_2$	$\mu_3 = f_1, f_2, f_1$	$\mu_4 = f_1, f_1, w_3$
$\mu_5 = f_1, w_2, w_3$	$\mu_6 = f_1, w_2, f_1$	$\mu_7 = f_1, w_2, f_2$	$\mu_8 = f_1, f_2, w_3$
$\mu_9 = f_2, f_2, f_1$	$\mu_{10} = f_2, f_1, f_1$	$\mu_{11} = f_2, f_1, f_2$	$\mu_{12} = f_2, f_2, w_3$
$\mu_{13} = f_2, w_2, w_3$	$\mu_{14} = f_2, w_2, f_2$	$\mu_{15} = f_2, w_2, f_1$	$\mu_{16} = f_2, f_1, w_3$
$\mu_{17} = w_1, w_2, f_2$	$\mu_{18} = w_1, f_1, w_3$	$\mu_{19} = w_1, f_2, w_3$	$\mu_{20} = w_1, f_1, f_1$
$\mu_{21} = w_1, f_2, f_2$	$\mu_{22} = w_1, f_1, f_2$	$\mu_{23} = w_1, f_2, f_1$	$\mu_{24} = w_1, w_2, f_1$
$\mu_{25} = w_1, w_2, w_3$			

Consider the following firms' preferences over the set of all feasible matching:

$P^*_{f_1} = \mu_6, \mu_3, \mu_4, \mu_1, \mu_{20}, \mu_{10}, \mu_7, \mu_5, \mu_8, \mu_2, \mu_{22}, \mu_{16}, \mu_{18}, \mu_{11}, \mu_{25}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_{21}, \mu_{23}, \mu_{24}, \mu_{15}, \mu_9.$

$P^*_{f_2} = \mu_{14}, \mu_{11}, \mu_{21}, \mu_2, \mu_{12}, \mu_9, \mu_7, \mu_{22}, \mu_{17}, \mu_1, \mu_{13}, \mu_{16}, \mu_{15}, \mu_{10}, \mu_{19}, \mu_8, \mu_{23}, \mu_3, \mu_{25}, \mu_5, \mu_{20}, \mu_6, \mu_{18}, \mu_{24}, \mu_4.$

And the following workers' preferences over the set of firms and the prospect of remaining unmatched:

$P_{w_1} = f_2, f_1, w_1.$

$P_{w_2} = f_2, f_1, w_2.$

$P_{w_3} = f_1, f_2, w_3.$

As in the previous example, we consider that firms have a set of estimation functions that satisfies *full admissibility* and block a matching according to our notion of stability in the presence of externalities, i.e. they compare the current matching with all admissible matchings after deviating. On the other hand, workers only make a comparison between their current and prospect partners when they plan to deviate from a current matching.

In order to illustrate the problem, consider the case of the matching $\mu_3 = f_1, f_2, f_1$. It is easy to show that μ_3 can be blocked by the coalition $\{f_2, \{w_1\}\}$. First of all, note that for the firm f_2 the set of all admissible matchings after deviating is given by $A(f_2, \{w_1\}) = \{\mu_{10}, \mu_{13}, \mu_{15}, \mu_{16}\}$. In addition, according to f_2's preferences it is easy to observe that $\mu' P^*_{f_2} \mu_3$ for all $\mu' \in A(f_2, \{w_1\})$. On the other hand, note that according to w_1's preferences $f_2 P_{w_1} \mu_3(w_1)$. Hence, the matching μ_3 cannot be φ-stable. By a similar argument, it is easy to show that every feasible matching can be blocked by either a coalition or individual worker. In fact, it is not difficult to show that every feasible matching of the problem can be blocked by either a worker or a coalition as we show in the following table.

Table 4: Blocking coalitions

Matching	Blocked by
$\mu_1 = f_1, f_1, f_2$	$\{f_2, \{w_1, w_3\}\}$
$\mu_{10} = f_2, f_1, f_1$	$\{f_2, \{w_1, w_2\}\}$
$\mu_{11} = f_2, f_1, f_2$	$\{f_1, \{w_2, w_3\}\}$
$\mu_2 = f_1, f_2, f_2$	$\{f_2, \{w_1, w_3\}\}$
$\mu_9 = f_2, f_2, f_1$	$\{f_1, \phi\}$

In addition, any matching that leaves w_1 unmatched is blocked by either $\{f_1, \{w_1\}\}$ or $\{f_2, \{w_1\}\}$. Any matching that leaves w_2 unmatched is blocked by either $\{f_1, \{w_2\}\}$, $\{f_2, \{w_2\}\}$ or $\{f_2, \{w_2, w_3\}\}$. Finally, any matching that leaves w_3 unmatched is blocked by $\{f_2, \{w_1, w_3\}\}$. Then as in the previous example, the set of φ-stable matchings of this problem is empty, i.e. $\mathcal{E}(F, W, P^*, q) = \emptyset$.

Sasaki and Toda (1996) analyze the marriage problem with externalities and introduce this notion of stability based on the concept of estimation functions. As in our more general setting, they also show that a φ-stable matching may not exists given particular sets of *estimation functions*. However, in addition they also provide a set of estimations that guarantees such existence. Further, they claim that a φ-stable matching exists if and only if the set of *estimation functions* satisfies *full admissib*ility. Hafalir (2008) shows that this conjecture is incorrect, since he characterizes a set of endogenous estimations that does not satisfies *full admissibility* and guarantees the existence of φ-stable matchings.

Our examples follow this line and show that a φ-stable matching may not exists under particular sets of estimation functions. However, our results are even more general, since we also show that in this setting the condition of full admissibility is not sufficient to assure the existence of φ-stable matchings. The following result gives even more insights about the scope of these results, since they allow us to establish more general conclusions about the existence of φ-stable assignments in many-to-one matching problems.

Lemma 1 *Let (F, W, P^*, q) be any instance of the matching problem with externalities. If there are no φ-stable matchings under full admissibility, then there are no φ-stable matchings for any possible set of estimation functions.*

Note that Examples 1 and 2 and Lemma 1 imply the following general impossibility result.

Theorem 1 *In many-to-one matching problems with externalities, no set of estimation functions ′ guarantees the existence of φ-stable matchings.*

The previous result has crucial implications for the analysis of matching problems with externalities. On the one hand, our result shows that for any the set of estimation functions there exist at least one matching problem with no φ-stable matchings. On the one hand, this result suggests that a restriction on the domain of agents' preferences seems necessary to guarantee the existence of φ-stable assignments.

2.3 THE EXISTENCE OF φ-STABLE MATCHINGS UNDER *full admissibility*

In the previous section, we argue that the impossibility of finding a set of estimation functions that guarantees the existence of φ-stable matchings suggests that a restriction on the domain of preferences may be necessary to guarantee the existence of such allocations. The analysis of standard matching problems arises to similar conclusions, since the existence of stable matchings in many-to-one problems can be guaranteed only under restricted domains of preferences (Roth and Sotomayor,1992).

The substitutability of preferences (Kelso and Crawford, 1982) and several generalizations of this condition[4] have been the main way to guarantee the existence of stable matchings in many-to-one problems. However, these restrictions on agents' preferences may not be well defined in matching problems with externalities. First of all, note that firms may not be able to define their optimal choices. In the presence of externalities, agents' preferences depend on the complete matching between firms and workers, hence a firm may not be able to choose the best group of workers among a set of candidates without considering the matching of the rest of agents. This argument implies that a generalization of the condition of substitutability requires a careful treatment in the presence of externalities.

In the rest of this section, we focus on matching problems with estimation functions that satisfy full admissibility. Under these conditions, we introduce a restriction on agents' preferences called *bottom q-substitutability* that generalizes the condition of q-substitutability (Cantala, 2004) to environments with externalities and guarantees the existence of φ-stable matchings. We also consider a very interesting approach to analyze the problem that consists in constructing a *reduced problem* without externalities that allows us to apply standard results to establish the existence of φ-stable matchings.[5]

2.3.1 *The reduced problem*

In this section, we show how to construct a consistent *reduced problem* for any matching problem with externalities. For this purpose, we require some additional notation. Given a set of *estimation functions* **'**, for each firm $f \in F$ (worker $w \in W$) the matching μ_f^S (μ_w^a) satisfies the following conditions: a) $\mu_f^S \in \varphi_f(S)$ ($\mu_w^a \in \varphi_w(a)$) and b) $\mu' R_f^* \mu_f^S$ for all $\mu' \in \varphi_f(S)$ ($\mu'' R_w^* \mu_w^a$ for all $\mu'' \in \varphi_w(a)$). Then μ_f^S (μ_w^a) is the least preferred admissible matching where the firm f (worker w) is matched with a feasible set of workers $S \in H_f$ (with an agent $a \in F \cup \{w\}$). Note that each of these least preferred

4 For instance, the condition of q -substitutability introduced by Cantala (2004).

5 Obviously, this approach is useful if the reduced problem is well defined and when the existence of stable matching in reduced problem leads us to establish the existence of φ-stable matchings in the original problem with externalities. Examples of this approach can be found in Shapley and Shubik (1969), Sasaki and Toda (1996), Hafalir (2008) and Klaus and Klijn (2005).

matchings is well defined, since agents' preferences are strict and complete and by definition the *estimations functions* map nonempty subsets of matchings.

Given any matching problem (F, W, P^*, q) and a set of *estimation functions* ', we define for each firm $f \in F$ a preference relation P_f^{φ} over the set of feasible sets of workers H_f in the following way: for any two different subsets of workers $S, S' \in H_f$, the preference relation satisfies $SP_f^{\varphi}S'$if and only if $\mu_f^S P_f^* \mu_f^{S'}$. In a similar way, for each worker $w \in W$ and any pair of different agents $a, a' \in F \cup \{w\}$, the preference relation P_w^{φ} satisfies $aP_w^{\varphi}a'$ if and only if $\mu_w^a P_w^* \mu_w^{a'}$.

Since the preference order P_a^* is complete, strict and transitive for each agent $a \in F \cup W$, we know that P_a^{φ} is well defined in the sense that is a complete, strict and transitive preference relation over the set of agents on the opposite side of the market. Hence, each matching problem with externalities (F, W, P^*, q) induces a well defined matching problem without externalities denoted by (F, W, P^{φ}, q) where $P^{\varphi} = \left(P_{f_1}^{\varphi}, ..., P_{f_m}^{\varphi}; P_{w_1}^{\varphi}, ..., P_{w_n}^{\varphi}\right)$ denotes the profile of agents' preferences. For each agent $a \in F \cup W$, let R_a^{φ} denote the weak preference relation associated with P_a^{φ}.

Example 3 *Consider the matching problem already introduced in the Example 2. Take, for instance, the feasible subsets of workers $\{w_1, w_3\}$ and $\{w_1, w_2\}$ for the firm f_1. It is easy to show that the least preferred matchings associated with these subsets of workers are $\mu_{f_1}^{\{w_1,w_3\}} = \mu_3$ and $\mu_{f_1}^{\{w_1,w_2\}} = \mu_1$. According to f_1's preferences $\mu_3 P_{f_1}^* \mu_1$, this induces a preference relation that satisfies $\{w_1, w_3\} P_{f_1}^{\varphi} \{w_1, w_2\}$. It is easy to construct the whole profile of preferences that characterizes the reduced problem of this example:*

$$P_{f_1}^{\varphi} = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \phi, \{w_3\}.$$

$$P_{f_2}^{\varphi} = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \phi.$$

We make the same exercise for the matching problem already introduced in Example 2. In this case, we have the following profile of preferences:

$$P_{f_1}^{\varphi} = \{w_1, w_2\}, \phi, \{w_1\}, \{w_2\}, \{w_3\}, \{w_2, w_3\}, \{w_1, w_3\}.$$

$$P_{f_2}^{\varphi} = \{w_3\}, \{w_1\}, \{w_2\}, \phi.$$

$$P_{w_1}^{\varphi} = f_1, f_2, w_1.$$

$$P_{w_2}^{\varphi} = f_2, f_1, w_2.$$

$$P_{w_3}^{\varphi} = f_1, w_3, f_2.$$

As we argue before, our notion of stability for matching problem with externalities induces a natural notion of stability for problems without externalities. Such notion of stability depends on the following definitions.

Definition 6 *A matching μ is blocked by a worker $w \in W$ if $wP_w\mu(w)$.*

Definition 7 *A coalition firm-set of workers $\{f, S\}$ such that $S \in H_f$ and $\mu(f) \neq S$, blocks the matching μ if:*

1. $SP_f\mu(f)$ *and*

2. $fR_w\mu(w)$ *for all* $w \in S$.

A matching μ is stable in the matching problem (F, W, P, q), if it is not blocked by any worker or coalition. Let $\mathcal{E}(F, W, P, q)$ denote the set of stable matchings of the problem.

The following result is crucial to characterize the existence φ-stable matchings, it basically says that any φ-admissible matching that is stable matching in the *reduced problem* is also φ-stable in the original problem.

Proposition 8 *Let* (F, W, P^*, q) *be any matching problem with externalities and* ' *any set of estimation functions. Then any matching* μ *that satisfies: a)* $\mu \in \varphi_a(\mu(a))$ *for all* $a \in F \cup W$ *and b)* $\mu \in \mathcal{E}(F, W, P^\varphi, q)$ *is* φ*-stable, i.e.* $\mu \in \mathcal{E}_\simeq(F, W, P^*, q)$.

The previous result has a crucial implication, since every matching is φ-admissible under *full admissibility* then any stable matching of the *reduced problem* is also φ-stable. We establish this observation as the following Corollary of the previous result.

Corollary 1 *Assume that the set of estimation functions* '*, satisfies full admissibility then* $\mathcal{E}(F, W, P^\varphi, q) \subset \mathcal{E}(F, W, P^*, q)$.

Note that the converse of the previous result does not necessarily hold as we show in the following example.

Example 4 *Consider a matching problem with three workers* $W = \{w_1, w_2, w_3\}$ *and two firms* $F = \{f_1, f_2\}$ *with quotas* $q_{f_1} = q_{f_2} = 2$. *Firms have the same preferences of the matching problem in Example 2, while workers' preferences are the following,*

$P^*_{w_1} = \mu_9, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{10}, \mu_1, \mu_8, \mu_7, \mu_4, \mu_5, \mu_6, \mu_3, \mu_2, \mu_{17}, \mu_{23}, \mu_{24}, \mu_{25}, \mu_{22}, \mu_{18}, \mu_{21}, \mu_{19}, \mu_{20}.$

$P^*_{w_2} = \mu_{23}, \mu_{21}, \mu_{19}, \mu_{12}, \mu_9, \mu_8, \mu_3, \mu_2, \mu_{22}, \mu_{20}, \mu_{18}, \mu_{16}, \mu_{11}, \mu_4, \mu_1, \mu_{10}, \mu_{25}, \mu_{24}, \mu_{17}, \mu_{15}, \mu_{13}, \mu_7, \mu_6, \mu_5, \mu_{14}.$

$P^*_{w_3} = \mu_{24}, \mu_{23}, \mu_{20}, \mu_{15}, \mu_9, \mu_6, \mu_3, \mu_{10}, \mu_{22}, \mu_{21}, \mu_{17}, \mu_{14}, \mu_{11}, \mu_7, \mu_1, \mu_2, \mu_{25}, \mu_{19}, \mu_{18}, \mu_{16}, \mu_{13}, \mu_8, \mu_5, \mu_4, \mu_{12}.$

As in previous examples, we consider a set of estimation functions that satisfies full admissibility. Under this assumption, it is not difficult to show that the *reduced problem* of this example is characterized by the following profile of preferences:

$P^\varphi_{f_1} = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \phi, \{w_3\}.$

$P^\varphi_{f_2} = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \phi.$

$P^\varphi_{w_1} = f_2, f_1, w_1.$

$P^\varphi_{w_2} = f_2, f_1, w_2.$

$P^\varphi_{w_3} = f_1, f_2, w_3.$

Note that the *reduced problem* (F, W, P^{φ}, q) of this example has no stable matchings, i.e. $\mathcal{E}(F, W, P^{\varphi}, q) = \varnothing$. However, it is also easy to show that the matching $\mu_{11} = f_2, f_1, f_2$ is φ-stable.

In the following section, we introduce a restriction over agents' preferences called *bottom q-substitutability* that generalizes the condition of *q-substitutability* to matching problems with externalities and, under *full admissibility,* assures the existence of φ-stable matchings.

2.3.2 *Bottom q-substitutability*

The condition of q-substitutability (Cantala, 2004) is a restriction on firms preferences that generalizes the well known condition of substitutability (Kelso and Crawford, 1982) to matching problems where firms have quotas of workers[6]. The condition of q-substitutability guarantees the existence of stable matchings in standard many-to-one matching problems. According to Proposition 1 (and Corollary 1) the existence of φ-stable matchings can be established by imposing conditions on the matching problem that assure the existence of stable matchings in the *reduced problem*. In this section, we introduce a restriction on agents' preferences called *bottom q-substitutability* that generalizes the condition of *q-substitutability* to matching problems with externalities.

Before introducing this condition, we require some additional notation. For each firm $f \in F$ and any subset of feasible workers $S \in H_f$, the matching $\mu_{f,S}$ satisfies the following conditions: a) $\mu_{f,S} \in A(f, S)$ and b) $\mu' R^*_f \mu_{f,S}$ for all $\mu' \in A(f, S)$. In a similar way, for each worker $w \in W$ and any agent $a \in F \cup \{w\}$, the matching $\mu_{w,a}$ satisfies the following conditions: a) $\mu_{w,a} \in A(w, a)$ and b) $\mu' R^*_w \mu_{w,a}$ for all $\mu' \in A(w, a)$. Note that the matching $\mu_{f,S}$ ($\mu_{w,a}$) is the f's (w's) least preferred matching where the firm f (the worker w) and the subset of feasible workers $S \in H_f$ (and the agent $a \in F \cup \{w\}$) are matched. Note that these matchings are well defined, since agents' preferences are strict and complete for all agents. Let $M_f = \{\mu \in \mathcal{M} : \mu = \mu_{f,S} \text{ and } S \in H_f\}$ denote the set of "least preferred matchings for the firm f" . Given the previous notation, we are able to introduce the following definition.

Definition 8 *For each firm $f \in F$ and any subset of workers $S \in 2^W$, the mapping $\Upsilon_f : 2^W \to \mathcal{M}$ satisfies the following conditions:*

1. $\Upsilon_f(S) \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$; *and*
2. $\Upsilon_f(S) R^*_f \mu'$ *for all* $\mu' \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$.

The mapping Υ_f can be interpreted as the choice function of the firm f in the presence of externalities. This mapping is defined under a min-max argument where for any given subset of workers, firms choose the best matching among the worse possible assignments. Note that the choice function Υ_f is well defined, since the preference relation P^*_f is strict and complete for each firm $f \in F$ and the set of

6 Let P_f be a preference relation of the firm f. The mapping $Ch_f : 2^W \to H_f$ denotes the optimal choice of the firm f. For any subset of workers $S \in 2^W$, the f's choice function satisfies the following conditions: a) $Ch_f(S) \in H_f$ and b) $Ch_f(S) R_f S'$ for all $S' \subset S$. Given any problem (F, W, P, q), we say that f's preferences are q-substitutable if for any subset of workers $S \in 2^W$ such that $w, w' \in S$ and $w \neq w'$, if $w \in Ch_f(S)$ then $w \in Ch_f(S \setminus \{w'\})$.

matchings $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f \subset \mathcal{M}$ is always nonempty. For instance, it is clear that the matching $\mu_{f,\emptyset}$ belongs to this set $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$, since by definition $\mu_{f,\emptyset}(f) = \emptyset \subset S$ for any subset of workers $S \subset W$. It is also clear that $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f = M_f$ whenever $S = W$, which implies that the choice function $Y_f(W)$ maps the most preferred matching among the ones in the set M_f. This choice function allows us to introduce a notion of substitutability for matching problems with externalities.

Definition 9 *We say that the preference profile P^* in the problem (F, W, P^*, q) satisfies the condition of bottom $q-$substitutability if for every firm $f \in F$ and any set of workers $S \in 2^W$ such that $w, w' \in S$, if $w \in \mu(f)$ such that $Y_f(S) = \mu$ implies that $w \in \mu'(f)$ such that $Y_f(S \setminus \{w'\}) = \mu'$.*

Note that for each firm f and any feasible set of workers $S \in H_f$, there exists a unique matching in the set M_f that satisfies $\mu(f) = S$. Hence, under full admissibility the profile of preferences of the *reduced problem* is fully characterized by the restricted preference relation over the set of least preferred matchings M_f. Thus, under full admissibility the profile of preferences P^* of the matching problem (F, W, P^*, q) is *bottom $q-$substitutable* if and only if the profile of preferences P^φ of the reduced problem (F, W, P^φ, q) is *$q-$substitutable.* Let $\mathcal{BS}$ denote the set of all preference profiles that satisfy the condition of *bottom q-substitutability.* Then we establish the following result.

Theorem 2 *Let (F, W, P^*, q) be any matching problem with externalities. Suppose that $P^* \in \mathcal{BS}$, then under full admissibility the set of φ-stable matchings $\mathcal{E}(F, W, P^*, q)$, is not empty.*

The previous result implies that the condition of *bottom q-substitutability* is sufficient to assure the existence of φ-stable matchings but not necessary. Consider, for instance, the following matching problem already introduced in Example 4. This matching problem induces a reduced problem with the following agents' preferences:

$P^\varphi_{f_1} = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \phi, \{w_3\}.$

$P^\varphi_{f_2} = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \phi.$

$P^\varphi_{w_1} = f_2, f_1, w_1.$

$P^\varphi_{w_2} = f_2, f_1, w_2.$

$P^\varphi_{w_3} = f_1, f_2, w_3.$

According to these preferences, the firm f_1 should choose the set of workers $\{w_1, w_3\}$ from the set $\{w_1, w_3\}$. However, when only the worker w_3 is available, the firm f_1 prefers to choose the empty. This simple example implies that the firm f_1 is not willing to substitute the worker w_1 with the worker w_3. Hence, this profile of preferences is not *q-substitutable*, which implies that the profile of preferences of the original problem P^* is not *bottom q-substitutable.* Further, we already argue that the set of stable matchings of this reduced problem is empty. However, under full admissibility the matching $\mu_{11} = f_2, f_1, f_2$ is φ-stable.

In the following example, we analyze a problem with *bottom q-substitutable* preferences that guarantees the existence of at least one φ-stable matching. In order to clarify all previous concepts and notation, we analyze the problem with some detail.

Example 5 *Consider a matching problem with two firms $F = \{f_1, f_2\}$ with quotas quotas $q_{f_1} = q_{f_2} = 2$ and three workers $W = \{w_1, w_2, w_3\}$. The set of feasible matchings of this problem is given in the Table 3 of Example 2. Agents' preferences are given by the following lists:*

$P^*_{f_1} = \mu_6, \mu_4, \mu_5, \mu_8, \mu_1, \mu_7, \mu_2, \mu_{20}, \mu_{24}, \mu_{23}, \mu_{15}, \mu_{18}, \mu_{22}, \mu_{16}, \mu_{11}, \mu_{25}, \mu_{21}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$

$P^*_{f_2} = \mu_{14}, \mu_{13}, \mu_{21}, \mu_{16}, \mu_{12}, \mu_9, \mu_{15}, \mu_2, \mu_{17}, \mu_{19}, \mu_{22}, \mu_{23}, \mu_7, \mu_{10}, \mu_1, \mu_8, \mu_3, \mu_{25}, \mu_{24}, \mu_{20}, \mu_{18}, \mu_6, \mu_5, \mu_4, \mu_{11}.$

$P^*_{w_1} = \mu_1, \mu_2, \mu_3, \mu_5, \mu_6, \mu_7, \mu_8, \mu_4, \mu_9, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{10}, \mu_{17}, \mu_{18}, \mu_{19}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}, \mu_{20}.$

$P^*_{w_2} = \mu_2, \mu_8, \mu_9, \mu_{12}, \mu_{19}, \mu_{21}, \mu_{23}, \mu_3, \mu_1, \mu_4, \mu_{11}, \mu_{16}, \mu_{18}, \mu_{20}, \mu_{22}, \mu_{10}, \mu_5, \mu_6, \mu_7, \mu_{13}, \mu_{15}, \mu_{17}, \mu_{24}, \mu_{25}, \mu_{14}.$

$P^*_{w_3} = \mu_2, \mu_7, \mu_{11}, \mu_{14}, \mu_{17}, \mu_{21}, \mu_{22}, \mu_1, \mu_3, \mu_6, \mu_9, \mu_{15}, \mu_{20}, \mu_{23}, \mu_{24}, \mu_{10}, \mu_4, \mu_5, \mu_8, \mu_{13}, \mu_{16}, \mu_{18}, \mu_{19}, \mu_{25}, \mu_{12}.$

As we argue before, to check if the condition of bottom q-substitutability is satisfied, it is enough to consider the restricted firms' preferences on the sets of matchings M_{f_1} and M_{f_2}:

$\widetilde{P}^*_{f_1} = \mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$

$\widetilde{P}^*_{f_2} = \mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}.$

It is easy to show that these preferences are bottom q-substitutable. Formally, we need to check the firms' choices for any possible subset of workers and establish the condition. For instance, consider the case of the firm f_1 and the set of workers $S = \{w_1, w_2, w_3\}$. According to f_1's preferences, the following holds.

$Y_{f_1}(S) = \mu_1$ *with* $\mu_1(f_1) = \{w_1, w_2\}$;
$Y_{f_1}(S \setminus \{w_1\}) = \mu_{11}$ *with* $\mu_{11}(f_1) = \{w_2\}$;
$Y_{f_1}(S \setminus \{w_2\}) = \mu_2$ *with* $\mu_2(f_1) = \{w_1\}$ *and*
$Y_{f_1}(S \setminus \{w_3\}) = \mu_1$ *with* $\mu_1(f_1) = \{w_1, w_2\}$.

However, according to our previous arguments we can also check whether the reduced problem (F, W, P^φ, q) has q-substitutable preferences. It is easy to show that the reduced problem of this example has the following profile of preferences:

$P^\varphi_{f_1} = \{w_1, w_2\}, \{w_1\}, \{w_2\}, \phi, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}.$

$P^\varphi_{f_2} = \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\}, \phi, \{w_1, w_3\}.$

$P^\varphi_{w_1} = f_1, f_2, w_1.$

$P^\varphi_{w_2} = f_2, f_1, w_2.$

$P^{\varphi}_{w_3} = f_2, f_1, w_3.$

It is clear that these preferences are q-substitutable, then a φ-stable matching exists. It is easy to show that the matching μ_2 is stable in the reduced problem (F, W, P^{φ}, q). Hence, by Corollary 1 the matching μ_2 is also φ-stable, i.e. $\mu_2 \in \mathcal{E}(F, W, P^*, q)$.

2.4 PESSIMISTIC AGENTS.

As we describe before, full admissibility is a situation where every matching is considered admissible for every agent. On the one hand, this kind of estimation functions seem to be compatible with the absence of information about others' expectations, which leads agents to estimate that any matching may be considered admissible. On the other hand, under full admissibility not only all matchings are considered admissible but also each of the least preferred assignments of the problem. This implies that under under *full admissibility* agents are *pessimistic,* in the sense that they consider admissible the worse feasible matchings of the problem. Sasaki and Toda (1996) show that this kind of pessimistic expectations may be crucial to characterize the existence of stable assignments in matching problems with externalities.[7] In this section, we introduce a set of pessimistic estimation functions that rationalizes the expectations of pessimistic agents and allow us to characterize the existence of φ-stable matchings.

Formally, agents are *pessimistic* whenever the set of *estimation functions* φ satisfies the following conditions: 1) For each $f \in F$, $\mu_{f,S} \in \varphi_f(S)$ for all $S \in H_f$ and 2) for all $w \in W$, $\mu_{w,a} \in \varphi_w(a)$ for all $a \in F \cup w$. Under the assumption that agents are *pessimistic*, it is possible to establish the following result.

Proposition 9 *Assume that agents are pessimistic, then any stable matching of the reduced problem $\mu \in \mathcal{E}(F, W, P^{\varphi}, q)$, is not blocked by any coalition or individual worker in the matching problem (F, W, P^*, q).*

Note that the existence of φ-stable matchings does not come from the previous result, since the conditions of this argument do not imply the existence of φ-admissible matchings.

2.4.1 *Pessimistic estimation functions*

In order to rationalize the estimation functions of pessimistic agents, we consider the following intuitive argument. Suppose that a firm f is planning to be matched with some feasible group of workers $S \in H_f$. When agents are *pessimistic*, the matching $\mu_{f,S} \in A(f, S)$ is admissible by definition, i.e. $\mu_{f,S} \in \varphi_f(S)$. However, this firm f cannot consider admissible another matching $\mu \in A(f, S) \setminus \{\mu_{f,S}\}$ that may be blocked by some coalition, i.e. a matching for which there exists a coalition $\{f', S'\} \subset F \cup W \setminus \{f\}$ such that $S' \in H_{f'}$, whose preferences satisfy $\mu_{f',S'} P^*_{f'} \mu$ and $\mu_{w',f'} P^*_{w'} \mu$ for all $w' \in S'$. Intuitively, this firm f is able to anticipate that the matching $\mu \in A(f, S)$ will be eventually blocked by the coalition $\{f', S'\}$. Formally, the set of pessimistic estimation functions is characterized by the following conditions,

7 Note that Sasaki and Toda (1996)'s non-existence argument depends on a simple example where all agents but one are pessimistic.

Definition 10 *A matching μ is admissible for the firm f, i.e. $\mu \in \rho_f(\mu(f))$ if there is no coalition $\{f', S'\} \subset F \cup W \setminus \{f\}$ such that $S' \in H_{f'}$ that satisfies:*

1. $\mu_{f',S'} P^*_{f'} \mu$ *and*
2. $\mu_{w',f'} P^*_{w'} \mu$ *for all $w' \in S'$;*

and no subset of workers $S'' \subset W$ that satisfies:

1. $\mu_{w',w'} P^*_{w'} \mu$ *for all $w' \in S''$.*

In a similar way,

Definition 11 *A matching μ is admissible for the worker w, i.e. $\mu \in \rho_w(\mu(w))$ if there is no coalition $\{f, S\} \subset F \cup W \diagdown \{w\}$ that satisfies:*

1. $\mu_{f,S} P^*_f \mu$ *and*
2. $\mu_{w,f} P^*_w \mu$ *for all $w \in S$;*

and no subset of workers $S'' \in W \diagdown \{w\}$ such that:

1. $\mu_{w',w'} P^*_{w'} \mu$ *for all $w' \in S''$.*

Let $\text{æ} = \{(\rho_f(\cdot), \rho_w(\cdot)) : f \in F, w \in W \text{ and } P^*\}$ denote the set of *pessimistic estimation functions*. Note that this *estimations* depend on agents' preferences and rationalize the conjecture that all agents are *pessimistic* expectations. In general, the set of *pessimistic estimations functions* **æ** does not satisfy the condition of *full admissible* as we show in the following example.

Example 6 *Consider the matching problem with externalities already introduce in Example 5. The set of least preferred feasible matchings of the firms f_1 and f_2 are:*

$$M_{f_1} = \{\mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9\}$$
$$M_{f_2} = \{\mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}\}$$

Consider the following set of feasible matchings $A(\{f_1, \{w_1, w_2\}\}) = \{\mu_1, \mu_4\}$. By assumption the matching μ_1 is considered admissible by the firm f_1, i.e. $\mu_1 \in \rho_{f_1}(\{w_1, w_2\})$. However, the matching $\mu_4 = f_1, f_1, w_3$ can be blocked by the coalition $\{f_2, w_3\} \subset F \cup W \setminus \{f_1\}$, since $\mu_{f_2,\{w_3\}} = \mu_1$ and $\mu_{w_3,\{f_2\}} = \mu_1$ and according to f_2 and w_3 preferences,

1. $\mu_1 P^*_{f_2} \mu_4$ *and*
2. $\mu_1 P^*_{w_3} \mu_4$.

Hence, the matching $\mu_4$cannot be admissible for the firm f_1, i.e. $\mu_4 \notin \rho_{f_1}(\{w_1, w_2\})$. Similarly, consider the set of feasible matchings $A(w_1, f_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$. As in the previous case, by assumption the matching μ_4 is admissible for the firm f_1, i.e. $\mu_4 \in \rho_{w_1}(f_1)$. However, the matching $\mu_5 = f_1, w_2, w_3$ cannot be admissible, since the coalition $\{f_2, \{w_2, w_3\}\} \subset F \cup W \setminus \{f_2\}$ has incentives to block it. It is easy to show that, $\mu_{f_2,\{w_2,w_3\}} = \mu_2$, $\mu_{w_2,\{f_2\}} = \mu_3$ and $\mu_{w_2,\{f_2\}} = \mu_1$. And according to f_2, w_2 and w_3 preferences, the following is satisfied,

1. $\mu_2 P^*_{f_2} \mu_5$,
2. $\mu_3 P^*_{w_2} \mu_5$ *and*
3. $\mu_1 P^*_{w_3} \mu_5$.

Thus $\mu_5 \notin \rho_{w_1}(f_1)$.

The set of pessimistic estimation functions of this problem is given by the following sets of admissible matchings for each agent.

For the firm f_1***:***

$\rho_{f_1}(\{w_1, w_2\}) = \{\mu_1\}$;
$\rho_{f_1}(\{w_1, w_3\}) = \{\mu_3\}$;
$\rho_{f_1}(\{w_2, w_3\}) = \{\mu_{10}\}$;
$\rho_{f_1}(\{w_1\}) = \{\mu_2, \mu_7\}$;
$\rho_{f_1}(\{w_2\}) = \{\mu_{11}, \mu_{16}, \mu_{22}\}$;
$\rho_{f_1}(\{w_3\}) = \{\mu_9, \mu_{15}, \mu_{23}\}$ *and*
$\rho_{f_1}(\phi) = \{\mu_{12}, \mu_{13}, \mu_{14}, \mu_{17}, \mu_{19}, \mu_{21}\}$.

For the firm f_2*:*

$\rho_{f_2}(\{w_1, w_2\}) = \{\mu_9, \mu_{12}\}$;
$\rho_{f_2}(\{w_1, w_3\}) = \{\mu_{11}\}$;
$\rho_{f_2}(\{w_2, w_3\}) = \{\mu_2\}$;
$\rho_{f_2}(\{w_1\}) = \{\mu_{10}, \mu_{15}, \mu_{16}\}$;
$\rho_{f_2}(\{w_2\}) = \{\mu_3, \mu_8\}$;
$\rho_{f_2}(\{w_3\}) = \{\mu_1, \mu_7\}$ *and*
$\rho_{f_2}(\phi) = \{\mu_4, \mu_5, \mu_6, \mu_{18}, \mu_{20}, \mu_{24}\}$.

For the worker w_1*:*

$\rho_{w_1}(f_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_7\}$;
$\rho_{w_1}(f_2) = \{\mu_{10}, \mu_{11}, \mu_{12}, \mu_{15}, \mu_{16}\}$; *and*
$\rho_{w_1}(w_1) = \{\mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}\}$.

For the worker w_2***:***

$\rho_{w_2}(f_1) = \{\mu_1, \mu_{10}, \mu_{16}, \mu_{22}\}$;
$\rho_{w_2}(f_2) = \{\mu_2, \mu_3, \mu_8, \mu_{12}\}$ *and*
$\rho_{w_2}(w_2) = \{\mu_7, \mu_{13}, \mu_{14}\}$.

For the worker w_3***:***

$\rho_{w_3}(f_1) = \{\mu_3, \mu_9, \mu_{10}, \mu_{15}, \mu_{23}\}$;
$\rho_{w_3}(f_2) = \{\mu_1, \mu_2, \mu_7, \mu_{11}\}$ *and*
$\rho_{w_3}(w_3) = \{\mu_8, \mu_{12}, \mu_{19}\}$.

Note that according to these estimations neither of the following matchings:

$\{\mu_4, \mu_5, \mu_6, \mu_8, \mu_9, \mu_{11}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{17}, \mu_{16}, \mu_{18}, \mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}\}$,

is ρ*-admissible.*

As we already show in Example 5, the matching $\mu_2 = f_1, f_2, f_2$ should be stable in the *reduced problem* $(F, W, P^{æ}, q)$ associated with this example. According to Proposition 2, the matching $\mu_2 = f_1, f_2, f_2$ cannot be blocked by any worker or coalition in the original matching problem with externalities (F, W, P^*, q), since all agents are pessimistic. Further, note also that the matching μ_2 is ρ-admissible, since $\mu_2 \in \rho_a(\mu_2(a))$ for all $a \in F \cup W$. This last observation implies that the matching μ_2 is a ρ-stable matching, i.e. $\mu_2 \in \mathcal{E}_\rho(F, W, P^*, q)$. The following result shows that this characteristic holds in general. In particular, we show that given the set of *pessimistic estimations functions* **æ**, the condition of *bottom q-substitutability* is sufficient to assure the existence of ρ-stable matchings.

Theorem 3 *Given the set of pessimistic estimation functions* **æ**, *if $P^* \in \mathcal{BS}$ then the set of ρ-stable matchings, $\mathcal{E}_\rho(F, W, P^*, q)$ is not empty.*

According to the previous result, the set of pessimistic estimation functions not only rationalize the set of matchings that pessimistic agents consider admissible but also guarantees that every stable matching of the reduced problem will be admissible for all agents. In contrast to the case of full admissibility, not all matchings are required to be admissible in order to characterized the existence of φ-stable matchings, then the previous example is also sufficient to establish the following results.

Proposition 10 *The condition of full admissibility is neither necessary nor sufficient to assure the existence of φ-stable matchings in many-to-one matching problems with externalities.*

2.5 THE CORE

The core is the set of matchings not blocked by any coalition. Sasaki and Toda (1996) introduce a notion of the core for marriage problems with externalities. According to this notion of the core, once externalities are considered not only the set of φ-stable matchings and the core do not coincide but also the core may be empty. This results contrast to the case of standard marriage problem where the core and the set of stable matching always coincide.

Sasaki and Toda (1996)'s core may be empty, since in general deviating agents do not take into account the set of matchings that they consider admissible. Under this notion of the core, members of coalitions deviate even when they could be worse off under admissible matchings after deviating. We propose an alternative notion of the core by assuming that agents always consider their *estimation functions* when they plan to deviate as members of any coalition. Formally,

Definition 12 *A coalition $A \subset F \cup W$ blocks the matching μ, whenever there is another matching $\widehat{\mu} \neq \mu$ such that:*

1. *$\widehat{\mu}(a) \subset A$ for all $a \in A$;*
2. *$\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(\widehat{\mu}(f))$ and all $f \in A$; and*
3. *$\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(\widehat{\mu}(w))$ and all $w \in A$.*

The core given the set of *estimation functions* ′, or simply the φ-core, is the set of φ-admissible matchings not blocked by any coalition $A \subset F \cup W$. Let $\mathcal{C}_{\varphi}(F, W, P^*, q)$ denote the φ-core. The following result provides an interesting property of the φ-core.

Theorem 4 *Let (F, W, P^*, q) be any matching problem with externalities. Let* ′ *be any set of estimation functions, then $\mathcal{E}_{\varphi}(F, W, P^*, q) = \mathcal{C}_{\varphi}(F, W, P^*, q)$.*

The previous result has some crucial implication about the existence and properties of the core for matching problems with externalities. First of all, it is clear that all existence results and properties of the set of φ-stable matchings extent to the φ-core. And secondly, the equivalence between the set of φ-stable matchings and the φ-core does not depend on the set of estimation functions.

2.6 CONCLUSIONS

In this chapter, we analyze the existence of stable matchings in matching problems with externalities. We argue that standard results about the existence of stable matchings cannot be trivially extended. First of all, agents' preferences should be defined over the set of all feasible matchings of the problem instead of the set of agents on the opposite side of the market. Once externalities are considered, agents should anticipate the reaction of the rest of agents in the face of potential deviations.

We extend the notion of stability proposed by Sasaki and Toda (1996) based on the concept of *estimation functions*. The set of *estimation functions* of a given problem represents, a belief about the set of matchings that agents consider admissible given a conjecture about the reactive behavior of agents. In general, the set of *estimation functions* is not uniquely defined and may be exogenous given.

We show that in many to one problems, a φ-stable matchings may not exist. Further, it is possible to find instances of the problem with no φ-stable matchings for any given set of *estimations*. This impossibility theorem contrasts with previous results in marriage problems with externalities, where there exists at least one set of *estimations* that assures the existence of φ-stable matchings. Given this impossibility result, we focus on the case of *full admissibility*. In this case, we provide a restriction on firms' preferences called *bottom q-substitutability* that guarantees the existence of at least one φ-stable matching.

Under the assumption that agents are *pessimistic*, it is possible to construct a set of *pessimistic estimations* that rationalizes the set of estimation functions with pessimistic agents. The set of *pessimistic estimations* depends on agents' preferences and guarantees the existence of φ-stable matchings, providing preferences are *bottom q-substitutable*. In addition, our results show that the assumption of *full admissibility* is neither necessary nor sufficient for assuring the existence of φ-stable matchings.

Finally, we analyze the existence of the core. In previous literature is introduced a notion of the core which may be empty for some instances of the problem. We propose a notion of the core where agents consider their estimation functions called φ-core. We show that the set of φ-stable matchings and the φ-core coincide given any set of estimations ′. As a final remark, it is clear that all results extent to the marriage problem with externalities, since this is a particular case of the many-to-one matching problem.

3 A SIMPLE DECENTRALIZED MATCHING MECHANISM IN MARKETS WITH COUPLES

3.1 INTRODUCTION

Two main features characterize most of real life matching markets. First of all, most of these markets are decentralized, i.e. the outcomes of these mechanisms depend on the actions of individual decision-makers. Secondly, around the world the number of couples searching jointly for a job has been increased in recent years (Klaus, Klijn and Massó, 2007).

This chapter deals with the analysis of a simple decentralized matching mechanisms in job markets with couples. In this setting, we analyze a mechanism called **One Application Mechanism (**$\mathcal{OA}$), already introduced in the context of college admissions problems[1] (Alcalde, Pérez-Castrillo and Romero-Medina, 1998).

Recent literature[2] suggests that stable matchings may be expected equilibrium outcomes of simple decentralized job markets with individual decision makers. However, once the presence of couples is considered, it is not clear whether only stable matchings should be expected as equilibrium outcomes. A matching market with couples has crucial differences with respect to markets with individuals. Members of couples are still individual decision makers but they must aggregate their individual preferences in order to compare different pairs of job places.

In this chapter, we fully characterize the set of Subgame Perfect Equilibrium (SPE) outcomes of the game induced by the $\mathcal{OA}$, under the assumption that couples' preferences are responsive. We show that any stable matching of the market can be achieved as a SPE outcome of this game, but in contrast to the case of college admissions problems unstable matchings may be supported in SPE. However, we prove that only one special kind of instability is reasonable in equilibrium, and furthermore we show that this kind of instability of equilibrium outcomes comes from coordination failures between members of couples.

Our main result shows that the $\mathcal{OA}$ implements in SPE the set of pairwise stable matchings of markets with couples. Finally, our characterization provides evidence that decentralized matching mechanisms work very well and better than centralized mechanisms for some instances of the problem, since the the National Resident Matching Program (NRMP) is manipulable and may cycle around unstable matchings when couples preferences are responsive (Klaus, Klijn and Massó, 2007).

The rest of the chapter is organized as follows. In Section 3.2, we describe the basic model; in Section 3.3, we analyze the matching mechanism; in Section 3.4, we compare decentralized and centralized matching procedures; in Section 3.5, we present some conclusions. All proofs are in the appendix C.

1 See Roth and Sotomayor (1990) for a good survey on two-sided matching literature.
2 For instance, Alcalde, Pérez-Castrillo and Romero-Medina (1998); Alcalde and Romero-Medina (2000); Triossi (2009) and Haeringer and Wooders (2011).

3.2 THE MODEL

Let $H = \{h_1, ..., h_n\}$ be a finite set of n hospitals and let $S = \{s_1, ..., s_m\}$ be a finite set of m medical students. Assume that each hospital has only one job position and $|H| = |S| \geq 2k$ for any $k \geq 1$. A couple is an unordered pair of students, we denote a typical couple by a pair $c = (s_k, s_l)$ such that $s_k, s_l \in S$ and $s_k \neq s_l$. We assume that there is no single student, so each medical student is member of one couple. Let $C = \{c_1, ..., c_k\}$ denote the set of couples in the market and let $\emptyset$ and u be the hospital's prospect of having its position unfilled and the student's prospect of being unemployed, respectively. Let $\mathcal{H} = \{H \cup \{u\} \times H \cup \{u\}\} \diagdown \{(h, h) : h \in H\}$ denote the set of all possible ordered pairs of hospitals and the prospect of being unemployed. A typical element of the set $\mathcal{H}$ is denoted by (h_p, h_q).

Each hospital $h \in H$ has a strict, transitive and complete preference relation P_h on the set of medical students and the prospect of having a position unfilled, $S \cup \{\emptyset\}$. Let R_h denote the weak preference relation induced by P_h. So for any pair of medical students $s, s' \in S$, the preference relation sR_hs' implies either sP_hs' or $s = s'$. Let $P^H = \{P_h\}_{h \in H}$ denote the preference profile of hospitals. Each couple $c \in C$ has a strict, transitive and complete preference relation P_c over the set of pairs of job positions $\mathcal{H}$, as before we denote by R_c the weak preference relation induced by P_c. Let $P^C = \{P_c\}_{c \in C}$ denote the preference profile of couples. In the following, we fix the sets of hospitals and medical students H and S, so a market with couples is completely described by a two-tuple (P^H, P^C).

In order to simplify, we focus on the simplest one to one matching problem. Formally, a matching between hospitals and medical students is defined as follows,

Definition 13 *A **matching** μ is a mapping from $H \cup S$ into $H \cup S$ such that:*

1. *for all $s \in S$, if $\mu(s) \notin H$ implies that $\mu(s) = u$;*
2. *for all $h \in H$, if $\mu(h) \notin S$ implies that $\mu(h) = \emptyset$; and*
3. *$\mu(s) = h$ if and only if $\mu(h) = s$.*

Let $M(P^H, P^C)$ denote the set of all feasible matchings in the market (P^H, P^C). Now we introduce the usual notion of stability in markets with couples. First of all, we introduce the concept of **individually rational** matchings,

Definition 14 *A matching μ is **individually rational** if,*

1. *for all $c = (s_k, s_l) \in C$, $(\mu(s_k), \mu(s_l)) R_c (\mu(s_k), u)$, $(\mu(s_k), \mu(s_l)) R_c (u, \mu(s_l))$ and $(\mu(s_k), \mu(s_l)) R_c (u, u)$; and*
2. *for all $h \in H$, $\mu(h) R_h \emptyset$.*

Secondly, the we introduce the concept of **blocking coalitions**,

Definition 15 *The coalition $(c = (s_k, s_l), (h_p, h_q))$ blocks the matching μ if,*

1. *$(h_p, h_q) P_c (\mu(s_k), \mu(s_l))$ and*
2. *$h_p \in H$ implies $s_k R_{h_p} \mu(h_p)$ and $h_q \in H$ implies $s_l R_{h_q} \mu(h_q)$.*

A matching μ is **stable** if it is individually rational and not blocked by any coalition. Let $S\left(P^H, P^C\right) \subset M\left(P^H, P^C\right)$ denote the set of stable matchings of the market $\left(P^H, P^C\right)$.

It is well known that the set of stable matchings may be empty in matching markets with couples (Roth and Sotomayor, 1990). So we assume that couples' preferences are **responsive**[3] in order to assure the existence of at least one stable matching in the problem (Klaus and Klijn, 2005; Klaus, Klijn and Nakamura, 2009). Formally,

Definition 16 *A couple $c = (s_k, s_l)$ has **responsive preferences** if there exists individual preferences $\succ_{s_k}$ and $\succ_{s_l}$such that for all $h_p, h_q, h_r \in H \cup \{u\}$,*

1. *$h_p \succ_{s_k} h_r$ implies $\left(h_p, h_q\right) P_c \left(h_r, h_q\right)$ and*
2. *$h_p \succ_{s_l} h_r$ implies $\left(h_q, h_p\right) P_c \left(h_q, h_r\right)$.*

If the preference orders $\succ_{s_k}$ and $\succ_{s_l}$exist, then they are unique.

Since the paper deals with a problem of implementation in markets with couples, we need a description of this tool. The literature on implementation theory is well known[4], so we briefly describe the notion of an extensive form matching mechanisms and the concept of implementation in Subgame Perfect Equilibrium (SPE). An **extensive form matching mechanism** (Triossi, 2009) is an array $G = \langle S \cup H, K, A, g \rangle$. $S \cup H$ is the set of players, K is the set of histories and A is the strategy space. Let Z be the set of terminal histories. Given the initial history k^0, any strategy profile $a \in A$ defines a unique terminal history $z_a \in Z$. Let $g : Z \to M\left(P^H, P^C\right)$ be the outcome function, this function specifies an outcome matching for each terminal history. A SPE is a strategy profile a^* that induces a Nash equilibrium of every subgame. Let $z_{a^*} \in Z$ be the terminal history induced by a SPE a^*, the matching $g\left(z_{a^*}\right)$ is called SPE outcome of the extensive for game induced by G, let $SPE\left(G\right)$ denote the set of SPE outcomes of G. Let $\mathcal{S}$ be the set of matching markets and let $\Phi : \mathcal{S} \twoheadrightarrow \Phi$ be a matching correspondence. An extensive form matching mechanism G implements Φ in SPE if for every market $\left(P^H, P^C\right) \in \mathcal{S}$, $SPE\left(G\right) = \Phi\left(P^H, P^C\right)$.

3.3 THE ONE APPLICATION MECHANISM

In this section, we analyze a very simple decentralized matching mechanisms already introduced in the context of college admissions problems (Alcalde, Pérez-Castrillo and Romero-Medina, 1998; Alcalde and Romero-Medina, 2000). Even when this mechanisms is very simple, it mimics many real life matching procedures and allows us analyze the strategic behavior of agents who face decentralized matching mechanisms.

This mechanism is called **One Application Mechanism** ($\mathcal{OA}$) and runs in the following two stages:

3 Weaker domains of preferences assure the existence of stable matchings. For instance, the weakly responsive preferences (Klaus and Klijn, 2005; Klaus, Klijn and Nakamura, 2009; Nakamura, 2005). However, all relevant results in couples markets, i.e. the existence of stable matchings, the lost of the lattice structure and the non-existence of the (hospital) student-optimal stable matching, hold under responsive preferences. So we restrict our analysis on this domain of preferences.

4 See Maskin, E. and Sjostrom, T. (2002).

1. **Application**: Each student $s \in S$ sends a message, $m(s) \in H \cup \{u\}$, where $m(s) = u$ implies that the student s prefers to remain unmatched and $m(s) \in H$ implies that s applies to some hospital $h \in H$. Let $M(h)$ denote the set of students who apply to the hospital h;

2. **Hiring**: Each hospital $h \in H$ considers its proposers $M(h)$ and the prospect of having its position unfilled $\emptyset$. Hospitals choose an element in the set $M(h) \cup \{\emptyset\}$. Let $J_h(M(h) \cup \{\emptyset\})$ denote the choice of the hospital h from the set $M(h) \cup \{\emptyset\}$.

Since students send at most one application and each hospital chooses at most one candidate, the outcome of the $\mathcal{OA}$ is a well defined matching. The $\mathcal{OA}$ induces a game in extensive form denoted by Γ^{OA}, where $S \cup H$ is the set of players and at each step of the game agents play simultaneously. In previous literature, it is shown that the $\mathcal{OA}$ implements in SPE the set of stable matchings of college admissions problems (Alcalde, Pérez-Castrillo and Romero-Medina, 1998; Alcalde and Romero-Medina, 2000).

We analyze the $\mathcal{OA}$ in the context of matching markets with couples, note that this extension is not direct. First of all, a couples market is fundamentally different to the standard matching problem, since even in the simplest setting (the one-to-one matching problem) the existence of stable matchings is not guaranteed (Roth and Sotomayor, 1990; Gale and Shapley, 1962). So, even in the simplest setting, we need to consider restrictions on couples' preferences to assure the existence of stable matchings.

Second, when agents face strategically matching mechanisms, they usually evaluate outcomes and possible deviations through individual preferences, thus the presence of couples introduces an additional problem. In a market with couples, we usually specify couples' preferences without considering individual preferences, since it is considered that individuals aggregate their preferences and only consider the welfare of the couple. Hence, we have to analyze a problem with individual strategic decision makers in a setting where agents evaluate outcomes (and possible deviations) through couples' preferences.

We introduce these characteristics of matching markets with couples in our analysis. First of all, we introduce some additional notation. For any given market with couples (P^H, P^C), let's define for each hospital $h \in H$, the h's choice function as follows.

Definition 17 *For any $S' \subset S$, the h's choice function $C_h : S' \cup \{\emptyset\} \to S' \cup \{\emptyset\}$ satisfies the following:*

1. $C_h(S' \cup \{\emptyset\}) \in S' \cup \{\emptyset\}$ *and*

2. $C_h(S' \cup \{\emptyset\})\, R_h x$ *for all* $x \in S' \cup \{\emptyset\}$.

We are interested in characterizing the set of SPE (in pure strategies) outcomes of the game Γ^{OA} induced by the mechanism. First of all, we consider the strategic behavior of hospitals. In this game, each hospital has a dominant strategy that coincides with the decision rule $J_h(\cdot) = C_h(\cdot)$, clearly this rule is optimal and independent on the strategies of all other agents. It is obvious that, at the last stage of the $\mathcal{OA}$ the best strategy for a hospital is to choose the best medical student among the available applicants. Clearly, at any SPE of the game Γ^{OA}, hospitals must follow their dominant strategy $C_h(\cdot)$.

Given the profile of optimal choice rules $J^* = \{C_h(\cdot)\}_{h \in H}$, the $\mathcal{OA}$ induces a n-players game in strategic form $G^{OA} = \left(S, \{H \cup \{s\}\}_{s \in S}, P^C\right)$ played by medical students. Note that the extensive form game Γ^{OA} has a SPE in pure strategies if and only if the associated strategic form game G^{OA} has a Nash equilibrium in pure strategies. Observe that under the $\mathcal{OA}$, any equilibrium in pure strategies of the game Γ^{OA} yields a well defined matching, otherwise students must play mixed strategies and choose a probability distribution over the set of actions $H \cup \{s\}$.

Before introducing our main results, we consider a simple example to show the difficulties of the problem that come from the strategic behavior of medical students. Consider the following market with couples $\left(P^H, P^C\right)$ with four hospitals and four students. There are two couples in the market: $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$. Hospitals' and couples' preferences are described in Table 5, assume that these preferences are completed to be strictly unemployment averse, i.e. for each couple $c \in C$ and all pair of hospitals $h_p, h_q \neq u$, it is satisfied $\left(h_p, h_q\right) P_c \left(h_p, u\right) P_c \left(u, u\right)$ and $\left(h_p, h_q\right) P_c \left(u, h_q\right) P_c \left(u, u\right)$.

Table 5: Agent Preferences

P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_1 = (s_1, s_2)$
s_4	s_4	s_2	s_2	(h_1, h_2)	(h_4, h_2)
s_2	s_3	s_3	s_4	(h_4, h_1)	(h_4, h_3)
s_1	s_2	s_1	s_1	(h_4, h_3)	(h_4, h_1)
s_3	s_1	s_4	s_3	(h_4, h_2)	(h_3, h_1)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_1, h_4)	(h_3, h_2)
				(h_1, h_3)	(h_3, h_4)
				(h_3, h_4)	(h_2, h_4)
				(h_3, h_1)	(h_2, h_1)
				(h_3, h_2)	(h_2, h_3)
				(h_2, h_3)	(h_1, h_2)
				(h_2, h_4)	(h_1, h_4)
				(h_2, h_1)	(h_1, h_3)
				.	.

In order to simplify, we describe a matching by a four entry vector that specifies the partner of each hospital in the order (h_1, h_2, h_3, h_4), so for instance, the matching $\mu = (s_1, s_2, s_3, s_4)$ implies that $\mu(h_1) = s_1$, $\mu(h_2) = s_2$ and so on. Note that according to Table 5, couples do not have responsive preferences, further the set of stable matchings of this market is empty. Under these conditions, we establish the next result.

Claim 1 *Consider the market with couples with preferences described in Table 5, then there is no Nash equilibrium in pure strategies of the game $G^{SH} = \left(S, \{H \cup \{s\}\}_{s \in S}, P^C\right)$ induced by the $\mathcal{OA}$.*

Claim 1 has an interesting implication, since it shows that no matching of the previous example can be sustained in SPE (in pure strategies). Since $G^{SH} = \left(S, \{H \cup \{s\}\}_{s \in S}, P^C\right)$

is a strategic form game, there has to exist at least one Nash equilibrium in mixed strategies with a proper specification of utility functions. However, the result implies that no matching of this problem can be considered a reasonable outcome of the market. In the following result, we show that this problem disappears when couples' preferences are responsive[5].

Proposition 11 *Let $\left(P^H, P^C\right)$ be a market with couples where couples' preferences are responsive, then any stable matching of the market can be achieved as a SPE outcome of the game induced by the $\mathcal{OA}$.*

This result is consistent with previous findings in the context of college admission problems[6]. However, a natural question is whether only stable matchings can be expected SPE outcomes of the $\mathcal{OA}$. The answer to this question is negative, as we show in the following example the mechanism may achieve unstable matchings in SPE.

Example 7 *Consider a 4x4 market with couples with $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$ with the following preferences,*

Table 6: Agent Preferences

P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_1 = (s_1, s_2)$
s_2	s_3	s_1	s_2	(h_3, h_1)	(h_4, h_2)
s_3	s_4	s_3	s_3	(h_1, h_2)	(h_4, h_3)
s_4	s_1	s_4	s_1	(h_4, h_1)	(h_4, h_1)
s_1	s_2	s_2	s_4	(h_2, h_1)	(h_3, h_2)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_3, h_2)	(h_3, h_4)
				(h_1, h_3)	(h_3, h_1)
				(h_4, h_2)	(h_1, h_2)
				(h_2, h_3)	(h_1, h_4)
				(h_3, h_4)	(h_1, h_3)
				(h_1, h_4)	(h_2, h_4)
				(h_4, h_3)	(h_2, h_1)
				(h_2, h_4)	(h_2, h_3)
				.	.

Couples' preferences in Table 6 are completed to be responsive. According to Proposition 1, stable matchings of this problem such as $\mu_9 = (s_2, s_3, s_1, s_4)$ and $\mu_{11} = (s_2, s_4, s_1, s_3)$ can be supported as SPE outcomes of the game Γ^{OA}.

We want to show that not only stable matchings can be achieved in SPE. For this purpose, consider that hospitals follow their dominant strategy, i.e. each $h \in H$ follows

5 Recall that responsiveness is sufficient, but not necessary to assure the existence of stable matching. The result of Proposition 1 is even more general and holds for any market $\left(P^H, P^C\right)$ where $S\left(P^H, P^C\right) \neq \emptyset$.

6 See, Alcalde, Pérez-Castrillo and Romero-Medina (1998), Alcalde and Romero-Medina (2000) and Triossi (2009).

the choice rule $C_h(\cdot)$. Let m be a profile of students' messages that satisfies: $m(s_1) = h_1$, $m(s_2) = h_3$, $m(s_3) = h_4$ and $m(s_4) = h_2$. Clearly, the outcome of this strategy profile is the matching $\mu_5 = (s_1, s_4, s_2, s_3)$.

We show that the profile of messages m is a Nash equilibrium of the game G^{SH} and as consequence a SPE of the game induced by the OA. Note that neither of the medical students s_3 and s_4 would like to deviate from the profile m, since the pair of job positions (h_4, h_2) is the top choice of the couple $c_2 = (s_3, s_4)$. Note also that for the student s_1, there is no $h_p \in H \cup \{u\}$ such that $(h_p, \mu_5(s_2)) \succ_{c_1} (\mu_5(s_1), \mu_5(s_2))$, hence there is no profitable deviation for s_1. Now consider the case of the student s_2. Note that only the hospital h_2 is a candidate for a profitable deviation, since $(\mu_5(s_1), h_2) \succ_{c_1} (\mu_5(s_1), \mu_5(s_2))$. However, if the student s_2 deviates with an alternative message $m'(s_2) = h_2$, the hospital h_2 (following the optimal choice rule $C_{h_2}(\cdot)$) will choose the medical student s_4. Hence, s_2 will be unmatched after deviating and $(\mu_5(s_1), \mu_5(s_2)) \succ_{c_1} (\mu_5(s_1), u)$ by responsiveness. Then, there is no profitable deviation for any student which implies that the matching $\mu_5 = (s_1, s_4, s_2, s_3)$ is a Nash equilibrium outcome of the game G^{SH} and by construction a SPE outcome of the game Γ^{OA}. However, note that the matching μ_5 is not stable, since it is blocked by the coalition $\{c_1 = (s_1, s_2), (h_3, h_1)\}$ which by blocking induces the matching $\mu_{11} = (s_2, s_4, s_1, s_3)$.

This feature of markets with couples contrasts with previous findings in college admission problems. However, the equilibrium outcomes of the $\mathcal{OA}$ can be unstable in a very particular fashion, since the matching $\mu_5 = (s_1, s_4, s_2, s_3)$ is blocked only by the coalition $\{c_1 = (s_1, s_2), (h_3, h_1)\}$. Note that this blocking coalition has a very particular structure, since the couple $c_1 = (s_1, s_2)$ and hospitals h_1 and h_3 are able to block the matching μ_5, because the members of the couple $c_1 = (s_1, s_2)$ can exchange their positions in a profitable way for all students and hospitals in this coalition. Hence, if members of the couple $c_1 = (s_1, s_2)$ coordinate their strategies such that s_1 applies to h_3 and s_2 applies to h_1, the outcome of the $\mathcal{OA}$ would be the matching $\mu_{11} = (s_2, s_4, s_1, s_3)$ which is stable. So μ_5 is a SPE outcome of the $\mathcal{OA}$ because there is a coordination failure between the members of the couple c_1. In the following result, we show that this feature of markets with couples is general, since only this kind of instability is admissible in SPE.

Proposition 12 *Let (P^H, P^C) be a market with couples whose couples' preferences are responsive, then any SPE outcome of the game induced by the $\mathcal{OA}$ is a matching that is either stable or blocked by some coalition of the form: $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$.*

In the rest of this section, we show that the instability of SPE outcomes of the $\mathcal{OA}$ is compatible with another well known notion of stability. First of all, we introduce the formal definition of the concept of pairwise stability,

Definition 18 *A matching μ is blocked by a pair (h_p, s_k) such that $h_p \in H \cup \{u\}$ and $s_k \in S$ if,*

1. $(h_p, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$ *and*
2. $s_k P_{h_p} \mu(h_p)$.

In a similar way, a matching μ is blocked by a pair (h_q, s_l) such that $h_q \in H \cup \{u\}$ and $s_l \in S$. A matching μ is **pairwise stable**, if it is individually rational and not blocked by any pair. It is clear that the notion of pairwise stability is weaker than the usual notion of stability in markets with couples. Note that any unstable matching that is blocked only by coalitions of the form $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$ is pairwise stable. This observation allows us to establish the following result.

Theorem 5 *Let (P^H, P^C) be a market with couples where couples' preferences are responsive, then the $\mathcal{OA}$ implements in SPE the set of pairwise stable matchings of the market.*

The proof of Theorem 1 comes directly from Propositions 1 and 2. This result has a crucial implication, since it shows that the possible instability of SPE outcomes of the $\mathcal{OA}$ is not so strong. Only instabilities that come from coordination failures between members of couples are reasonable when agents face strategically the $\mathcal{OA}$. Further, those instabilities are compatible with the notion of pairwise stability in matching problems.

Note that along the paper, we have not assumed any coordination among agents, since usually it is very difficult to justify this coordination in strategic environments. However, the coordination of actions between members of a couple is very reasonable and natural in many economic situations. So, it is not difficult to argue that only stable matchings will be reasonable equilibrium outcomes of the $\mathcal{OA}$.

3.4 DECENTRALIZED VS CENTRALIZED MECHANISMS

Our main result shows that only pairwise stable matchings can be expected equilibrium outcomes when agents face strategically simple decentralized matching mechanisms in markets with couples. This result contrasts with the case of centralized mechanisms. One of the most popular and analyzed centralized matching procedures is the National Resident Matching Program[7] (NRMP). Klaus, Klijn and Massó (2007) analyze the new NRMP algorithm which introduces the presence of couples in the market for new medical students. They show that the NRMP may achieve unstable matchings when couples preferences are responsive. To illustrate the problem consider a simple 4 by 4 matching problem with preferences given in Table 7.

Couples' preferences are completed to be responsive, this assumption implies that there exists at least one stable matching. Klaus, Klijn and Massó (2007) apply that new NRMP algorithm to the previous example and find that this algorithm cycles around the unstable matching $\mu = (\emptyset, s_3, s_4, \emptyset)$. Note that the matching μ is neither stable nor pairwise stable, since the coalition (s_1, h_1) blocks μ. According to Theorem 1, μ cannot be supported by any SPE under the $\mathcal{OA}$. If the stability of outcome matchings measures the success of a matching mechanism, our main result shows that decentralized matching mechanisms work very well and better than centralized mechanisms for some instances of matching markets with couples.

7 The National Resident Matching Program (NRMP) is a centralized mechanism where is applied an algorithm to match hospitals and medical students in the USA. The purpose of the NRMP is matching hospitals and physician in a stable way, in this algorithm the presence of couples has been explicitly incorporated. See Roth, A. (1984) and Roth, A. (2008) to have a clear idea about the importance of the market of new physicians in the theory of market design.

Table 7: Agent Preferences

P^H				P^C	
h_1	h_2	h_3	h_4	$c_1 = (s_1, s_2)$	$c_2 = (s_3, s_4)$
s_2	s_2	s_2	s_2	(h_1, h_2)	(h_2, h_3)
s_3	s_3	s_3	s_3	(h_1, h_3)	(h_2, h_4)
s_1	s_1	s_1	s_1	(h_1, h_4)	(h_2, h_1)
s_4	s_4	s_4	s_4	(h_2, h_1)	(h_1, h_3)
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(h_2, h_3)	(h_1, h_4)
				(h_2, h_4)	(h_1, h_2)
				(h_3, h_1)	(h_3, h_4)
				(h_3, h_2)	(h_3, h_2)
				(h_3, h_4)	(h_3, h_1)
				(h_4, h_1)	(h_4, h_3)
				(h_4, h_2)	(h_4, h_2)
				(h_4, h_3)	(h_4, h_1)
				.	.

3.5 CONCLUSIONS

We show that a simple decentralized matching mechanisms, like the $\mathcal{OA}$, implements in SPE the set of pairwise stable matchings of couples markets. In contrast with the NRMP algorithm, only pairwise stable matchings are expected SPE outcomes of the $\mathcal{OA}$. Given the usual notion of stability in couples markets, we show that the $\mathcal{OA}$ may attain unstable matchings in equilibrium. However, we show that this instability is very particular and comes exclusively from coordination failures between members of couples.

A

APPENDIX A: PROOFS OF CHAPTER 1

A.0.1 *The signaling equilibrium*

The maximization problem of any student with parameter α is:

$$\max_{P_1\geq 0}\left\{\sum_{k=1}^{M} v_k\binom{N-1}{k-1}F\left(\rho_N^{-1}\left(P_1\right)\right)^{N-k}\left[1-F\left(\rho_N^{-1}\left(P_1\right)\right)\right]^{k-1}-\frac{c\left(P_1\right)}{\phi\left(\alpha\right)}\right\} \tag{25}$$

Let's define the function $\varphi\left(x,N,k\right)=F\left(x\right)^{N-k}\left[1-F\left(x\right)\right]^{k-1}$. Hence, for each $k\in\{2,...,N-1\}$ it is satisfied the following,

$$\varphi'\left(x,N,k\right)=\left[\left(N-k\right)\left(1-F\left(x\right)\right)-\left(k-1\right)F\left(x\right)\right]F\left(x\right)^{N-1-k}\left[1-F\left(x\right)\right]^{k-2}f\left(x\right). \tag{26}$$

Hence, the FOC of the payoff function $\pi\left(\alpha,P_1\right)$ with respect to P_1 is given by,

$$\left\{\begin{array}{c} v_1\left(N-1\right)F\left(\rho_N^{-1}\left(P_1\right)\right)^{N-2}\frac{f\left(\rho_N^{-1}\left(P_1\right)\right)}{\rho_N'\left(\rho_N^{-1}\left(P_1\right)\right)}+\\ \sum\limits_{k=2}^{M}v_k(N-k)\binom{N-1}{k-1}F\left(\rho_N^{-1}\left(P_1\right)\right)^{N-1-k}\left[1-F\left(\rho_N^{-1}\left(P_1\right)\right)\right]^{k-1}\frac{f\left(\rho_N^{-1}\left(P_1\right)\right)}{\rho_N'\left(\rho_N^{-1}\left(P_1\right)\right)}-\\ -\sum\limits_{k=2}^{M}v_k\left(k-1\right)\binom{N-1}{k-1}F\left(\rho_N^{-1}\left(P_1\right)\right)^{N-k}\left[1-F\left(\rho_N^{-1}\left(P_1\right)\right)\right]^{k-2}\frac{f\left(\rho_N^{-1}\left(P_1\right)\right)}{\rho_N'\left(\rho_N^{-1}\left(P_1\right)\right)} \end{array}\right\}-\frac{c'\left(P_1\right)}{\phi\left(\alpha\right)}=0 \tag{27}$$

In a symmetric equilibrium it is satisfied $P_1=\rho_M\left(\alpha\right)$, then

$$\left\{\begin{array}{c} v_1\left(N-1\right)\phi\left(\alpha\right)F\left(\alpha\right)^{N-2}f\left(\alpha\right)+\\ +\sum\limits_{k=2}^{M}v_k\left(N-k\right)\binom{N-1}{k-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-1-k}\left[1-F\left(\alpha\right)\right]^{k-1}f\left(\alpha\right)-\\ -\sum\limits_{k=2}^{M}v_k\left(k-1\right)\binom{N-1}{k-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-k}\left[1-F\left(\alpha\right)\right]^{k-2}f\left(\alpha\right) \end{array}\right\}=c'\left(\rho_N\left(\alpha\right)\right)\rho_N'\left(\alpha\right) \tag{28}$$

By reordering and solving this differential equation with the initial condition $\rho_M\left(0\right)=0$, we find that the signaling strategy $\rho_M\left(\alpha\right)$ satisfies,

$$\rho_M\left(\alpha\right)=c^{-1}\left(\begin{array}{c} \left(N-1\right)\sum\limits_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\binom{N-2}{k-1}\int_0^\alpha\phi\left(x\right)F\left(x\right)^{N-1-k}\left[1-F\left(x\right)\right]^{k-1}f\left(x\right)dx+...\\ ...+\left(N-1\right)v_M\binom{N-2}{M-1}\int_0^\alpha\phi\left(x\right)F\left(x\right)^{N-M-1}\left[1-F\left(x\right)\right]^{M-1}f\left(x\right)dx \end{array}\right) \tag{29}$$

This completes the maximization problem of any student with parameter α. Note that $\left(N-1\right)\binom{N-2}{k-1}=\frac{(N-1)!}{(k-1)!(N-1-k)!}$, then we can re-write this signaling strategy as follows,

$$\rho_M\left(\alpha\right)=c^{-1}\left(\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^\alpha\phi\left(x\right)f_{(k,N-1)}\left(x\right)dx+v_M\int_0^\alpha\phi\left(x\right)f_{(M,N-1)}\left(x\right)dx\right) \tag{30}$$

Where $f_{(k,N-1)}(x) = \frac{(N-1)!}{(k-1)!(N-1-k)!} F(x)^{N-1-k}[1-F(x)]^{k-1} f(x)$ for $k = 1, ..., M$. Note that $f_{(k,N-1)}(x)$ is the density probability function of the x-th highest order statistic from an iid sample of size $N-1$.

Proof of proposition 2:

Consider that any student with parameter α is planing to deviate from the signaling strategy $\rho_M(\alpha)$ by choosing an alternative score P'. Assume w.l.g. that $0 \leq P' < \rho_M(\alpha)$, since the signaling strategy is strictly increasing in α there exists a unique $0 \leq \alpha' < \alpha$ such that $P' = \rho_M(\alpha')$. Then by choosing the score P' the student gets the expected payoff $\pi(\alpha, P') = \pi(\alpha, \rho_M(\alpha'))$ given by,

$$\pi\left(\alpha, \rho_M\left(\alpha'\right)\right) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1} - \frac{c\left(\rho_M\left(\alpha'\right)\right)}{\phi(\alpha)} \tag{31}$$

By deviating to P', the student losses the extra payoff,

$$\pi(\alpha, \rho_M(\alpha)) - \pi\left(\alpha, \rho_M\left(\alpha'\right)\right) = \begin{cases} \sum_{k=1}^{M} v_k \binom{N-1}{k-1}\left(F(\alpha)^{N-k}[1-F(\alpha)]^{k-1} - F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1}\right) - \\ -\frac{1}{\phi(\alpha)}\left(c(\rho_M(\alpha)) - c\left(\rho_M\left(\alpha'\right)\right)\right) \end{cases} \tag{32}$$

Note that the increment in the signaling cost $c(\rho_M(\alpha)) - c(\rho_M(\alpha'))$ is positive and can be written as,

$$c(\rho_M(\alpha)) - c\left(\rho_M\left(\alpha'\right)\right) = \sum_{k=1}^{M-1}(v_k - v_{k+1})\int_{\alpha'}^{\alpha}\phi(x) f_{(k,N-1)}(x)\,dx + v_M\int_{\alpha'}^{\alpha}\phi(x) f_{(M,N-1)}(x)\,dx \tag{33}$$

Since $\phi(x)$ is strictly increasing and positive in x, it is clear that

$$\frac{1}{\phi(\alpha)}\left(c(\rho_M(\alpha)) - c\left(\rho_M\left(\alpha'\right)\right)\right) < \sum_{k=1}^{M-1}(v_k - v_{k+1})\int_{\alpha'}^{\alpha} f_{(k,N-1)}(x)\,dx + v_M\int_{\alpha'}^{\alpha} f_{(M,N-1)}(x)\,dx \tag{34}$$

Note that by reordering the previous equation, we find that the following condition holds,

$$\frac{1}{\phi(\alpha)}\left(c(\rho_M(\alpha)) - c\left(\rho_M\left(\alpha'\right)\right)\right) < (N-1)\,v_1\int_0^{\alpha} F(x)^{N-2} f(x)\,dx + \sum_{k=2}^{M} v_k \binom{N-1}{k-1}\int_0^{\alpha}\varphi'(x, N, k)\,dx \tag{35}$$

Note that for any $k \in \{2, ..., N-1\}$, it holds

$$\int_{\alpha'}^{\alpha}\varphi'(x, N, k)\,dx = F(\alpha)^{N-k}[1-F(\alpha)]^{k-1} - F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1} \tag{36}$$

Which implies that $\pi(\alpha, \rho_M(\alpha)) - \pi(\alpha, \rho_M(\alpha')) > 0$. By a similar argument, it is possible to show that any deviation $P' > \rho_N(\cdot)$ cannot be a profitable deviation. This completes the proof.

Proof of Proposition 3:

- $\rho_M(\alpha)$ is strictly increasing in α and bounded above.

To prove that the signaling strategy $\rho_M(\alpha)$ is strictly increasing in α, it is enough to show that the function $c(\rho_M(\alpha))$ is strictly increasing in α, since $c(\cdot)$ is a strictly increasing function. Then

$$\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right) = \begin{cases} (N-1)\sum\limits_{k=1}^{M-1}\left(v_k - v_{k+1}\right)\binom{N-2}{k-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-1-k}\left[1-F\left(\alpha\right)\right]^{k-1}f\left(\alpha\right)+... \\ ...+(N-1)\,v_M\binom{N-2}{M-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-M-1}\left[1-F\left(\alpha\right)\right]^{M-1}f\left(\alpha\right) \end{cases} \quad (37)$$

It is clear that $\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right) > 0$ for all α, as we desired. To prove that signaling strategy $\rho_M\left(\alpha\right)$ is bounded above, we use the the fact that this function can be written as follows,

$$c\left(\rho_M\left(\alpha\right)\right) = \sum_{k=1}^{M-1}\left(v_k - v_{k+1}\right)\int_0^\alpha \phi\left(x\right)f_{(k,N-1)}\left(x\right)dx + v_M\int_0^\alpha \phi\left(x\right)f_{(M,N-1)}\left(x\right)dx \quad (38)$$

Where $f_{(k,N-1)}\left(x\right)$ is the density function of the $k-th$ order statistic from an $N-1$ sample with distribution function $F\left(x\right)$ such that $x_{(1,N-1)} = \underset{1\leq i\leq N-1}{max}\left\{x_i\right\}$, $x_{(2,N-1)}$=second greatest in $\left\{x_i\right\}_{i=1}^{N-1}$ and so on. Since $\phi\left(x\right)$ is strictly increasing and bounded in $[0,w]$, we know that

$$c\left(\rho_M\left(\alpha\right)\right) \leq \phi\left(w\right)\left(\sum_{k=1}^{M-1}\left(v_k - v_{k+1}\right)\int_0^w f_{(k,N-1)}\left(x\right)dx + v_M\int_0^w f_{(M,N-1)}\left(x\right)dx\right) \quad (39)$$

But by definition $\int_0^w f_{(k,N-1)}\left(x\right)dx = 1$ for all $k=1,...,M$. Then

$$c\left(\rho_M\left(\alpha\right)\right) \leq \phi\left(w\right)\left(\sum_{k=1}^{M-1}\left(v_k - v_{k+1}\right) + v_M\right) < \infty \quad (40)$$

- $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)$ is strictly increasing in α .

Now we want to show that the equilibrium payoff $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)$ is strictly increasing in α. To prove this property, we calculate the derivative of the payoff function with respect to α. Then

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right) = \begin{cases} (N-1)\,v_1F\left(\alpha\right)^{N-2}f\left(\alpha\right)+\sum\limits_{k=2}^{M}v_k\binom{N-1}{k-1}\varphi'\left(\alpha\right)- \\ -\frac{1}{\phi(\alpha)^2}\left(\phi\left(\alpha\right)\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right)-c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)\right) \end{cases} \quad (41)$$

By reordering the previous expression, it is easy to show that

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right) = \frac{c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)}{\phi\left(\alpha\right)^2} > 0 \quad (42)$$

This completes the proof.

A.0.2 *Comparative statics*

Proof of Proposition 4:

Let $\rho_M(\alpha, N)$ be the equilibrium signaling strategy of any college admissions problem with $M \geq 1$ colleges and $N > M$ students. Since the cost function $c(\cdot)$ is strictly increasing, it is enough to show that the function $c(\rho_M(\alpha, N))$ satisfies the desired properties. Then, it is easy to show that the difference $c(\rho_M(\alpha, N+1)) - c(\rho_M(\alpha, N))$ is equal to

$$\left\{ \begin{array}{l} \sum_{k=1}^{M-1} (v_k - v_{k+1}) \int_0^\alpha \left(N\binom{N-1}{k-1} F(x) - (N-1)\binom{N-2}{k-1} \right) \phi(x) F(x)^{N-1-k} [1-F(x)]^{k-1} f(x)\, dx + ... \\ \quad ... + v_M \int_0^\alpha \left(N\binom{N-1}{M-1} F(x) - (N-1)\binom{N-2}{M-1} \right) \phi(x) F(x)^{N-M-1} [1-F(x)]^{M-1} f(x)\, dx \end{array} \right. \tag{43}$$

Given that $(N-1)\binom{N-2}{k-1} = (N-k)\binom{N-1}{k-1}$, the previous equation reduces to the following,

$$\left\{ \begin{array}{l} \sum_{k=1}^{M-1} (v_k - v_{k+1}) \binom{N-1}{k-1} \int_0^\alpha (NF(x) - (N-k)) \phi(x) F(x)^{N-1-k} [1-F(x)]^{k-1} f(x)\, dx + ... \\ \quad ... + v_M \binom{N-1}{M-1} \int_0^\alpha (NF(x) - (N-M)) \phi(x) F(x)^{N-M-1} [1-F(x)]^{M-1} f(x)\, dx \end{array} \right. \tag{44}$$

Then, it is clear that $c(\rho_M(\alpha, N+1)) - c(\rho_M(\alpha, N)) < 0$ if $\alpha \leq \alpha_N(N)$ where $\alpha_N(N)$ is the unique solution to the equation,

$$F(x) = 1 - \frac{M}{N} \tag{45}$$

Further, the threshold $\alpha_N(N)$ is monotone increasing in N, i.e. $\alpha_N(N+1) > \alpha_N(N)$ for all $N > M$. This completes the proof.

Proof of Proposition 5:

Consider the equilibrium signaling strategy of any college admissions problem with one college with $M \geq 2$ school seats and $N \geq M+2$ students.

$$\rho_M(\alpha, M) = c^{-1}\left((N-1) v_1 \binom{N-2}{M-1} \int_0^\alpha \phi(x) F(x)^{N-1-M} [1-F(x)]^{M-1} f(x)\, dx \right) \tag{46}$$

Since the function $c(\cdot)$ is strictly increasing, to prove this result we focus on the function $c(\rho_M(\alpha, M))$. Then, it is easy to show that the difference $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M))$ is equal to

$$v_1(N-1) \int_0^\alpha \left(\binom{N-2}{M} (1-F(x)) - \binom{N-2}{M-1} F(x) \right) \phi(x) F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx \tag{47}$$

By reordering and applying the identity $\binom{N}{k} = \binom{N-1}{k-1} + \binom{N-1}{k}$, the previous equation reduces to the following,

$$v_1(N-1) \int_0^\alpha \left(\binom{N-2}{M} - \binom{N-1}{M} F(x) \right) \phi(x) F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx \tag{48}$$

Given that $\binom{N-1}{M} = \frac{N-1}{N-1-M}\binom{N-2}{M}$, we get

$$v_1(N-1) \binom{N-2}{M} \int_0^\alpha \left(1 - \frac{N-1}{N-1-M} F(x) \right) \phi(x) F(x)^{N-2-M} [1-F(x)]^{M-1} f(x)\, dx \tag{49}$$

Then, it is clear that $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M)) > 0$ if $\alpha \leq \alpha_N(M, N)$ where $\alpha_M(M, N)$ is the unique solution to the equation,

$$F(x) = 1 - \frac{N-1-M}{N-1} \tag{50}$$

Further, it is clear that the threshold $\alpha_M(M, N)$ is monotone increasing in N and monotone decreasing in M. This completes the proof.

Proof of Proposition 6:

We analyze the effect of a change in the quality of the college c_k, then we consider that the equilibrium signaling strategy depends on the quality of this college, i.e. $\rho_M(\alpha, v_k)$. We know that the equilibrium signaling strategy satisfies the equation,

$$c(\rho_M(\alpha, v_k)) = (N-1)v_1 \int_0^\alpha \phi(x) F(x)^{N-2} f(x)\, dx + \sum_{k=2}^{M} v_k \binom{N-1}{k-1} \int_0^\alpha \phi(x) \varphi'(x, N, k)\, dx \tag{51}$$

Consider a change in the quality of the college c_k such that $v_{k-1} > v'_k > v_{k+1}$, i.e. a change in college qualities that preserve students' ordinal preferences. It is not difficult to show that the difference $c(\rho_M(\alpha, v'_k)) - c(\rho_M(\alpha, v_k))$ satisfies the following equation for $k = 2, ..., M$,

$$c(\rho_M(\alpha, v'_k)) - c(\rho_M(\alpha, v_k)) = (v'_k - v_k) \binom{N-1}{k-1} \int_0^\alpha \phi(x) \varphi'(x, N, k)\, dx \tag{52}$$

Since $\varphi'(x, N, k) = [(N-k)(1-F(x)) - (k-1)F(x)] F(x)^{N-1-k} [1-F(x)]^{k-2} f(x)$, it is easy to show that whenever $\alpha \leq \alpha_{v_k}(N, k)$

$$sgn\left(c(\rho_M(\alpha, v'_k)) - c(\rho_M(\alpha, v_k))\right) = sgn(v'_k - v_k) \tag{53}$$

Where $\alpha_{v_k}(N, k)$ is the unique solution to the equation,

$$F(x) = \frac{N-k}{N-1} \tag{54}$$

Further, it is easy to observe that the threshold $\alpha_{v_k}(N, k)$ is monotone increasing in N for all $k = 2, ..., M$ and monotone decreasing in k. This completes the proof.

A.0.3 *Gains of the CSM*

If α is distributed according to an exponential distribution function $f(\alpha; \theta) = \frac{1}{\theta} e^{-\frac{\alpha}{\theta}}$ for $\alpha \in [0, \infty)$ and $\theta > 0$, then

1. $E[\alpha] = \theta$ and
2. $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1, ..., N$.

where $\alpha_{(1)} = \max_{1 \le i \le N} \alpha_i$, $\alpha_{(2)}$=second greatest in $\{\alpha_i\}_{i=1}^{N-1}$and so on (Huang, 1974). Consider any M by N college admissions problem such that $N > M \geq 1$, then colleges' gains satisfy,

$$\Delta EQ\left(j,N\right) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N}\right)^{j-1}. \tag{55}$$

Assume w.l.g. that $\theta = 1$. We establish the following auxiliary results.

Claim 2 *The continuous function $f\left(x\right) = \left(\frac{x-1}{x}\right)^{j-1}$ is strictly increasing and strictly concave in x for all $x > j \geq 3$.*

Proof. To prove this result, we simply take the first and second derivative of the function $f\left(x\right) = \left(\frac{x-1}{x}\right)^{j-1}$. Then it is easy to show the following:

1. $f'\left(x\right) = \left(\frac{j-1}{x^2}\right)\left(\frac{x-1}{x}\right)^{j-2} > 0$; and

2. $f''\left(x\right) = \left(\frac{j-1}{x^4}\right)\left(\frac{x-1}{x}\right)^{j-3}\left(j-2x\right) < 0$.

For all $x > j \geq 3$, this completes the proof. ■

Lemma 2 $\Delta EQ\left(j,N\right) > \Delta EQ\left(j+1,N\right)$ *for all* $j = 1,...,M-1$.

Proof. It is not difficult to show that the difference $\Delta EQ\left(j,N\right) - \Delta EQ\left(j+1,N\right)$ satisfies the following,

$$\Delta EQ\left(j,N\right) - \Delta EQ\left(j+1,N\right) = \frac{1}{j} - \left(\frac{N-1}{N}\right)^{j-1}\left(\frac{1}{N}\right) \tag{56}$$

Since, $\left(\frac{N-1}{N}\right)^{j-1} \leq 1$ for all $j \geq 1$, we know that

$$\Delta EQ\left(j,N\right) - \Delta EQ\left(j+1,N\right) \geq \frac{1}{j} - \frac{1}{N} = \frac{N-j}{jN}. \tag{57}$$

Since $N > M \geq j$, $\Delta EQ\left(j,N\right) - \Delta EQ\left(j+1,N\right) > 0$ for $j = 1,...,M-1$. This completes the proof. ■

Lemma 3 $\Delta EQ\left(j,N\right)$ *is strictly monotone increasing in* $N > M$ *for all* $j = 1,...,M$.

Proof. Consider the following function for a given $j = 1,...,M$,

$$\Delta EQ\left(j,N+1\right) - \Delta EQ\left(j,N\right) = \left\{\begin{array}{l} \sum_{k=1}^{N+2-j} \frac{1}{N+2-k} - \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \\ -\left[\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1}\right] \end{array}\right. \tag{58}$$

By simplifying, we can get the following expression,

$$\Delta EQ\left(j,N+1\right) - \Delta EQ\left(j,N\right) = \frac{1}{N+1} - \left(\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1}\right) \tag{59}$$

It is not difficult to show by a direct inspection that $\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > 0$ for $j = 1, 2$. Now consider the case of any j such that $N > M \geq j \geq 3$. By the Claim 1, we know that

$$f'(N) \geq \left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1} \tag{60}$$

where $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$ such that $x > j \geq 3$. Hence,

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) \geq \frac{1}{N+1} - \left(\frac{j-1}{N^2}\right)\left(\frac{N-1}{N}\right)^{j-2} \tag{61}$$

Since $\left(\frac{N-1}{N}\right)^{j-2} \leq 1$ for all $j \geq 2$, we know that

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) \geq \frac{1}{N+1} - \frac{j-1}{N^2} = \frac{N^2 - (j-1)\,(N+1)}{N^2\,(N+1)}. \tag{62}$$

Given that $N > M \geq j \geq 3$ and $(N-1)\,(N+1) > (j-1)\,(N+1)$, we conclude that

$$\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > \frac{1}{N^2\,(N+1)}. \tag{63}$$

Then $\Delta EQ\,(j, N+1) - \Delta EQ\,(j, N) > 0$ for all $N > M \geq j \geq 3$. This completes the proof. ■

Proof of Proposition 7

Properties 1 and 2 of colleges gains $\Delta EQ\,(j, N)$ come directly from Lemmas 1 and 2. For the third property, assume that $\Delta EQ\,(M, N) \geq 0$ for $N = M+1$, then $N^* = M+1$. By Lemma 2, $\Delta EQ\,(M, N) \geq 0$ for all $N \geq N^* > M$. By Lemma 1, $\Delta EQ\,(j, N) \geq 0$ for all $j = 1, ..., M$ provided $\Delta EQ\,(M, N) \geq 0$. Then $\Delta EQ\,(j, N) \geq 0$ for all $N \geq N^*$ and $j = 1, ..., M$.

Now suppose that $\Delta EQ\,(M, N) < 0$ for $N = M+1$. Note that,

1. $\lim_{N\to\infty} \left(\frac{N-1}{N}\right)^{M-1} = 1$ for all $M \geq 1$; and

2. $\lim_{N\to\infty} E\left[\alpha_{(M)}\right] = \lim_{N\to\infty} \sum_{k=1}^{N+1-M} \frac{1}{N+1-k} = \infty$.

Then there exists a $N^* > M$ such that $\Delta EQ\,(M, N^*) \geq 0$. Then by Lemmas 1 and 2, $\Delta EQ\,(j, N) \geq 0$ for all $N \geq N^*$ and $j = 1, ..., M$. This completes the proof.

B

APPENDIX B: PROOFS OF CHAPTER 2

Proof of Lemma 1:

Proof. Suppose that $\mathcal{E}(F, W, P^*, q) = \emptyset$, while $\mathcal{E}_{\varphi^*}(F, W, P^*, q) \neq \emptyset$ for an arbitrary set of *estimations* $'^*$. Consider any matching $\mu \in \mathcal{E}_{\varphi^*}(F, W, P^*, q)$, this matching is φ^*-admissible and not blocked by any worker or coalition given $'^*$. Since $\mathcal{E}(F, W, P^*, q) = \emptyset$, we have two cases under *full admissibility*:

Case 1: The matching μ is blocked by at least one coalition $\{f, S\}$ such that $S \in H_f$. Hence, it is satisfied 1. $\mu' P_f^* \mu$ for all $\mu' \in A(f, S)$ and 2. $\mu'' P_w^* \mu$ for all $\mu'' \in A(w, f)$ and all $w \in S$. By definition, $\varphi_f^*(S) \subset A(f, S)$ and $\varphi_w^*(f) \subset A(w, f)$ for all f and w in $F \cup W$. Then it is clear that a) $\mu' P_f^* \mu$ for all $\mu' \in \varphi_f^*(S)$ and b) $\mu'' P_w^* \mu$ for all $\mu'' \in \varphi_w^*(f)$ and all $w \in S$. A contradiction.

Case 2: The matching μ is blocked by some worker $w \in W$. Then $\mu(w) \neq w$ and $\mu' P_w^* \mu$ for all $\mu' \in A(w, w)$, this implies that $\mu' P_w^* \mu$ for all $\mu' \in \varphi_w^*(w)$. A contradiction, this completes the proof. ∎

Proof of Proposition 8:

Proof. Assume that the matching μ is blocked by some $\{f, S\}$ such that $S \in H_f$ in the problem (F, W, P^*, q) but stable in the reduced problem, i.e. $\mu \in \mathcal{E}(F, W, P^\varphi, q)$. If $\mu' P_f^* \mu$ for all $\mu' \in \varphi_f(S)$ implies that $\mu_f^S P_f^* \mu$, since $\mu \in \varphi_a(\mu(a))$ for all $a \in F \cup W$. We know that $\mu R_f^* \mu_f^{\mu(f)}$, hence $\mu_f^S P_f^* \mu_f^{\mu(f)}$ which implies $S P_f^\varphi \mu(f)$. Assume that there is some worker $w \in \mu(f) \cap S$, this implies that $\mu(w) = \{f\}$ and by assumption $\mu \in \varphi_w(\mu(w))$. Since μ is blocked by $\{f, S\}$, we know that $\mu'' P_{w'}^* \mu$ for all $\mu'' \in \varphi_{w'}(f)$ and all $w' \in S$. It is impossible that $\mu \in \varphi_w(f)$, hence $\mu(f) \cap S = \phi$. If $\mu'' P_w^* \mu$ for all $\mu'' \in \varphi_w(f)$ implies that $\mu_w^f P_w^* \mu$, since $\mu \in \varphi_a(\mu(a))$ for all $a \in F \cup W$. We know that $\mu R_w^* \mu_w^{\mu(w)}$, hence $\mu_w^f P_w^* \mu_w^{\mu(w)}$ which implies that $f P_w^\varphi \mu(w)$ for all $w \in S$. These conditions imply that $\mu(f) \neq S$, a) $S P_f^\varphi \mu(f)$ and b) $f P_w^\varphi \mu(w)$ for all $w \in S$, a contradiction.

Now assume that μ is blocked by an individual worker $w \in W$. In this case, we have that $\mu(w) \neq w$ and $\mu' P_w^* \mu$ for all $\mu' \in \varphi_w(w)$. By a similar argument this implies $\mu_w^w P_w^* \mu_w^{\mu(w)}$, hence $w P_w^\varphi \mu(w)$, a contradiction.

Since the matching $\mu \in \mathcal{E}(F, W, P^\varphi, q)$ is φ-admissible and not blocked by any worker or coalition, then μ is φ-stable. ∎

Proof of Theorem 2:

Proof. Assume that the condition of *full admissibility* holds. For each firm $f \in F$, we define the f's choice function as $Ch_f(S) = \mu(f)$ such that $\Upsilon_f(S) = \mu$ for any $S \subset W$. First, we have to show that for each $f \in F$ and any$S \subset W$ the choice function Ch_f maps the best subset of workers in S according to the preference relation P_f^φ. We know that for any$S \subset W$, $\Upsilon_f(S) \in M_f$ and $\Upsilon_f(S) R_f^* \mu'$ for all $\mu' \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$, then by definition $\mu(f) \subset S$ whenever $\Upsilon_f(S) = \mu$. Assume that there is some subset of workers $S' \subset S$, such that $S' P^{\varphi}_f \mu(f)$ for $\Upsilon_f(S) = \mu$. Hence, the preference relation $S' P^{\varphi}_f \mu(f)$ implies that $\mu^{S'}_f P^*_f \mu^{\mu(f)}_f$. By the condition of *full admissibility* $\Upsilon_f(S) = \mu = \mu^{\mu(f)}_f$, since $\mu^{S'}_f = \mu_{f,S'} \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$, a contradiction. Hence, $\mu(f) R^{\varphi}_f S'$ for all $S' \subset S$ where $\Upsilon_f(S) = \mu$. This implies that $Ch_f(S) = \mu(f)$ such that $\Upsilon_f(S) = \mu$ for any $S \subset W$ is a well defined choice function for each firm f.

Now we have to show that the preferences profile P^{φ} satisfies the condition of q-substitutability. Suppose that $w,\, w' \in S$ and $w \in \mu(f)$ where $\Upsilon_f(S) = \mu$, this implies that $w \in Ch_f(S)$. By *bottom q-substitutability,* we know that $w \in \mu'(f)$ where $\Upsilon_f(S \diagdown \{w'\}) = \mu'$, this implies that $w \in Ch_f(S \diagdown \{w'\})$, then the preferences profile P^{φ} satisfies q-substitutability. Hence, the *reduced problem* (F, W, P^{φ}, q) has a at least one stable matching. Since by *full admissibility* any feasible matching of the problem is φ-admissible, then $\mathcal{E}(F, W, P^*, q)$ is not empty. ■

Proof of Proposition 9:

Proof. Assume that the matching μ is stable in the *reduced problem* (F, W, P^{φ}, q) but blocked by some coalition $\{f, S\}$ such that $S \in H_f$ in the problem (F, W, P^*, q). Hence, $\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(S)$ implies that $\mu^S_f P^*_f \mu$ while $\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(f)$ implies $\mu^f_w P^*_w \mu$. By assumption, $\mu_{f,\mu(f)} = \mu^{\mu(f)}_f \in \varphi_f(\mu(f))$ and $\mu_{w,\mu(w)} = \mu^{\mu(w)}_w \in \varphi_w(\mu(w))$. Hence, $\mu R^*_f \mu^{\mu(f)}_f$ and $\mu R^*_w \mu^{\mu(w)}_w$ even if the matching μ is not φ-admissible. Then $\mu^S_f P^*_f \mu^{\mu(f)}_f$ and $\mu^f_w P^*_w \mu^{\mu(w)}_w$ for all $w \in S$, which imply that a) $S P^{\varphi}_f \mu(f)$ and b) $f P^{\varphi}_w \mu(w)$ for all $w \in S$, a contradiction.

Now suppose that μ is blocked by a worker $w \in W$. In a similar way, we have that $\mu(w) \neq w$ and $\mu' P^*_w \mu$ for all $\mu' \in \varphi_w(w)$ which implies $\mu^w_w P^*_w \mu^{\mu(w)}_w$. Hence, $w P^{\varphi}_w \mu(w)$, a contradiction. ■

Proof of Theorem 3:

Proof. Let $(F, W, P^{æ}, q)$ be the *reduced problem* associated with (F, W, P^*, q). By the condition of *bottom q-substitutability*, there exists at least one stable matching in the *reduced problem*, say $\mu^* \in \mathcal{E}(F, W, P^{æ}, q)$.

We have to show that μ^* is ρ-admissible. Suppose in contradiction that $\mu^* \notin \rho_f(\mu^*(f))$ for some firm $f \in F$. There are two cases:

Case 1: There exists a coalition $\{f', S'\} \subset F \cup W \diagdown \{f\}$ such that $S' \in H_{f'}$, $\mu_{f',S'} P^*_{f'} \mu$ and $\mu_{w',f'} P^*_{w'} \mu$ for all $w' \in S'$. Since agents are pessimistic, $\mu_{f',S'} = \mu^{S'}_{f'} \in \rho_{f'}(S')$ and $\mu_{w',f'} = \mu^{f'}_{w'} \in \rho_{w'}(f')$, then $\mu^{S'}_{f'} P^*_{f'} \mu^{\mu^*(f')}_{f'}$ and $\mu^{f'}_{w'} P^*_{w'} \mu^{\mu^*(w')}_{w'}$ for all $w' \in S'$. This implies that $S' P^{\rho}_{f'} \mu^*(f')$ and $f' R^{\rho}_{w'} \mu^*(w')$ for all $w' \in S'$, a contradiction.

Case 2: There exists a subset of workers $S'' \subset W$ such that $\mu_{w',w'} P^*_{w'} \mu$ for all $w' \in S''$. In a similar way as before, we know that $\mu_{w',w'} = \mu^{w'}_{w'} \in \rho_{w'}(w')$. Hence, $\mu^{w'}_{w'} P^*_{w'} \mu^{\mu^*(w')}_{w'}$ implies that $w' P^{\rho}_{w'} \mu^*(w')$ for all $w' \in S''$, a contradiction.

Given that f was any firm, this implies that $\mu^* \in \rho_f(\mu^*(f))$ for all $f \in F$. A similar argument applies for any worker, then the matching μ^* is ρ-admissible, i.e. $\mu^* \in \rho_a(\mu^*(a))$ for all $a \in F \cup W$. Hence, the matching μ^* is ρ-stable. This completes the proof. ■

Proof of Theorem 4:

Proof. Suppose that the matching $\mu \in \mathcal{E}_{\varphi}(F,W,P^{*},q)$ but $\mu \notin \mathcal{C}_{\varphi}(F,W,P^{*},q)$, then there is another matching $\widehat{\mu} \neq \mu$ and one coalition $A \subset F \cup W$ which blocks the matching μ. Take any firm $f \in A$ and the subset of worker $\widehat{\mu}(f) \subset A$, obviously $\widehat{\mu}(w) = \{f\} \subset A$ for all $w \in \widehat{\mu}(f)$. It is satisfied: 1) $\mu' P_{f}^{*} \mu$ for all $\mu' \in \varphi_{f}(\widehat{\mu}(f))$ and 2) $\mu'' P_{w}^{*} \mu$ for all $\mu'' \in \varphi_{w}(f)$ and all $w \in \widehat{\mu}(f)$. Then the coalition $\{f, \widehat{\mu}(f)\}$ blocks the matching μ. If there is no firm in the coalition A, take any worker $w \in A$, obviously $\widehat{\mu}(w) = \{w\} \subset A$ and it is satisfied that: 1) $\mu'' P_{w}^{*} \mu$ for all $\mu'' \in \varphi_{w}(w)$. Hence, any individual worker $w \in A$ blocks the matching μ, a contradiction.

On the other hand, suppose that $\mu \in \mathcal{C}_{\varphi}(F,W,P^{*},q)$ but $\mu \notin \mathcal{E}_{\varphi}(F,W,P^{*},q)$, then there is at least a coalition, $\{f,S\}$, or an individual worker, $w \in W$, which blocks the matching μ. Set the matching $\widehat{\mu}$, such that $\widehat{\mu}(f) = S$, obviously $\widehat{\mu} \neq \mu$, and $A = \{f,S\}$. By definition is satisfied: 1) $\mu' P_{f}^{*} \mu$ for all $\mu' \in \varphi_{f}(\widehat{\mu}(f))$ and all $f \in A$ and 2) $\mu'' P_{w}^{*} \mu$ for all $\mu'' \in \varphi_{w}(\widehat{\mu}(w))$ and all $w \in A$, then $\mu \notin \mathcal{C}_{\varphi}(F,W,P^{*},q)$. Suppose that and individual worker blocks the matching μ, set $A = \{w\}$ and $\widehat{\mu}(w) = w$, obviously $\widehat{\mu} \neq \mu$ and it is satisfied: 1)$\mu'' P_{w}^{*} \mu$ for all $\mu'' \in \varphi_{w}(\widehat{\mu}(w))$ and all $w \in A$, a contradiction. This completes the proof. ∎

C

APPENDIX C: PROOFS OF CHAPTER 3

Proof of Claim 1:

Proof. Each profile of pure strategies yields a matching, hence it is enough to show that for every profile of messages there is a profitable deviation for at least one student.

Consider any matching ν, where there is at least one unmatched agent. This implies that there is at least one unmatched student and a hospital with an unfilled position, say s and h. Let m be any profile of messages that yields the matching ν, and let $m(s)$ be the message of the student s. Since $\nu(h) = \emptyset$ and any student is acceptable for each hospital, we know that $M(h) = \varnothing$. Hence, the alternative message $m'(s) = h$ is a profitable deviation for s, since the hospital h follows its dominant strategy, i.e. $C_h(M'(h) \cup \emptyset) = s$ for $M'(h) = M(h) \cup \{s\}$. So, only matchings with no unmatched agents are candidates for equilibrium outcomes.

In the following list, we show the whole set of matchings with no unmatched agents. On the right of each matching, we show a student who has a profitable deviation given any profile of messages that yields each of these possible matchings:

Table 8: Profitable Deviations

Matching	Deviation	Matching	Deviation
$\mu_1 = (s_1, s_2, s_3, s_4)$	$m'(s_4) = h_2$	$\mu_{13} = (s_3, s_1, s_2, s_4)$	$m'(s_4) = h_2$
$\mu_2 = (s_1, s_2, s_4, s_3)$	$m'(s_4) = h_2$	$\mu_{14} = (s_3, s_1, s_4, s_2)$	$m'(s_2) = h_3$
$\mu_3 = (s_1, s_3, s_2, s_4)$	$m'(s_2) = h_4$	$\mu_{15} = (s_3, s_2, s_1, s_4)$	$m'(s_2) = h_4$
$\mu_4 = (s_1, s_3, s_4, s_2)$	$m'(s_4) = h_1$	$\mu_{16} = (s_3, s_2, s_4, s_1)$	$m'(s_2) = h_3$
$\mu_5 = (s_1, s_4, s_2, s_3)$	$m'(s_2) = h_4$	$\mu_{17} = (s_3, s_4, s_1, s_2)$	$m'(s_1) = h_1$
$\mu_6 = (s_1, s_4, s_3, s_2)$	$m'(s_4) = h_1$	$\mu_{18} = (s_3, s_4, s_2, s_1)$	$m'(s_2) = h_1$
$\mu_7 = (s_2, s_1, s_3, s_4)$	$m'(s_4) = h_1$	$\mu_{19} = (s_4, s_1, s_2, s_3)$	$m'(s_4) = h_2$
$\mu_8 = (s_2, s_1, s_4, s_3)$	$m'(s_4) = h_2$	$\mu_{20} = (s_4, s_1, s_3, s_2)$	$m'(s_2) = h_3$
$\mu_9 = (s_2, s_3, s_1, s_4)$	$m'(s_2) = h_4$	$\mu_{21} = (s_4, s_2, s_1, s_3)$	$m'(s_2) = h_4$
$\mu_{10} = (s_2, s_3, s_4, s_1)$	$m'(s_4) = h_1$	$\mu_{22} = (s_4, s_2, s_3, s_1)$	$m'(s_2) = h_3$
$\mu_{11} = (s_2, s_4, s_1, s_3)$	$m'(s_2) = h_4$	$\mu_{23} = (s_4, s_3, s_1, s_2)$	$m'(s_3) = h_3$
$\mu_{12} = (s_2, s_4, s_3, s_1)$	$m'(s_4) = h_1$	$\mu_{24} = (s_4, s_3, s_2, s_1)$	$m'(s_4) = h_4$

Then there is no Nash equilibrium in pure strategies. ■

Proof of Proposition 11:

Proof. Take any matching $\mu \in S(P^H, P^C)$, we know that there is at least one since couples' preferences are responsive. Consider a strategy profile such that:

1. $m(s) = \mu(s)$ for all $s \in S$; and
2. $J_h(\cdot) = C_h(\cdot)$ for all $h \in H$.

Given the profile $(m, J) = \left(\{\mu(s)\}_{s\in S}, \{C_h(\cdot)\}_{h\in H}\right)$, the outcome of the $\mathcal{OA}$ is the matching $g^{OA}(m, J) = \mu$. We show that there is no profitable deviation for any agent. We know that $C_h(\cdot)$ is an optimal decision rule for each hospital, then no hospital has a profitable deviation.

Consider any student $s \in S$, since μ is a stable matching we know that it is individually rational, then the alternative message $m'(s) = u$ is not a profitable deviation for any $s \in S$ such that $\mu(s) \in H$.

Suppose that $s \in S$, sends a message $m'(s_k) = h_p$ such that $(h_p, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$. Given that no other agent deviates, we know that $M'(h) = M(h)$ for all $h \neq \mu(s_k), h_p$. In addition, $M'(\mu(s_k)) = \varnothing$ and $M'(h_p) = M(h_p) \cup \{s_k\}$. Since μ is stable and $h_p \in H$, we know that $\mu(h_p)\, P_h s_k$, then $C_{h_p}(M'(h_p)) = \mu(h_p)$. Let ν be the outcome matching of the mechanism given the individual deviation $m'(s_k)$ then, $\nu(s_k) = u$ and $\nu(s_l) = \mu(s_l)$. Since μ is individually rational, it follows that $(\mu(s_k), \mu(s_l))\, R_c\, (u, \mu(s_l))$. Then, no students has a profitable deviation. ■

The next result is auxiliary in the characterization of the set of equilibrium outcomes of the $\mathcal{OA}$.

Lemma 4 *Suppose that couples preferences are responsive, then for each couple $c \in C$ and for all $h'_p, h'_q, h_p, h_q \in H \cup \{u\}$ such that $\left(h'_p, h'_q\right) \neq (h_q, h_p)$, if $\left(h'_p, h'_q\right) P_c\, (h_p, h_q)$ implies either $h'_p \succ_{s_k} h_p$ or $h'_q \succ_{s_l} h_q$.*

Proof. There are three cases:

1. Assume that $\left(h'_p, h'_q\right) \neq (h_q, h_p)$ and either $h'_p = h_q$ or $h'_q = h_p$. Consider without lost of generality (w.l.g.) that $h'_q = h_p$ and suppose that $\left(h'_p, h'_q\right) P_c\, (h_p, h_q)$ ($\left(h'_p, h_p\right) P_c\, (h_p, h_q)$ since $h'_q = h_p$). In contradiction, suppose that $h_p \succeq_{s_k} h'_p$ and $h_q \succeq_{s_l} h_p$. If $h_p = h'_p$ then $h'_p = u$ and $h_p \neq h_q$, hence $h_q \succ_{s_l} h_p$. By responsiveness, we know that $(h_p, h_q)\, P_c \left(h'_p, h_p\right)$, which is a contradiction. If $h_p \neq h'_p$ then $h_p \succ_{s_k} h'_p$ and by responsiveness $(h_p, h_q)\, P_c \left(h'_p, h_q\right)$ and $\left(h'_p, h_q\right) R_c \left(h'_p, h_p\right)$, hence $(h_p, h_q)\, P_c \left(h'_p, h_p\right)$ a contradiction.

2. Assume that $\left(h'_p, h'_q\right) P_c\, (h_p, h_q)$ and either $h'_p = h_p$ or $h'_q = h_q$. Suppose w.l.g. that $h'_q = h_q$, hence $h_q \succeq_{s_k} h'_q$. Consider in contradiction that $h_p \succ_{s_k} h'_p$, by responsiveness we have $(h_p, h_q)\, P_c \left(h'_p, h'_q\right)$, a contradiction.

3. Assume that $h'_p \neq h_q$, $h'_q \neq h_p$, $\left(h'_p, h'_q\right) \neq (h_q, h_p)$ and $\left(h'_p, h'_q\right) P_c\, (h_p, h_q)$. Suppose in contradiction that $h_p \succeq_{s_k} h'_p$ and $h_q \succeq_{s_l} h'_q$, note that both preferences have to be strict. Hence, by responsiveness $h_q \succ_{s_l} h'_q$ implies $\left(h'_p, h_q\right) P_c \left(h'_p, h'_q\right)$ and $h_p \succ_{s_k} h'_p$ implies $(h_p, h_q)\, P_c \left(h'_p, h_q\right)$. Hence $(h_p, h_q)\, P_c \left(h'_p, h'_q\right)$, a contradiction.

This completes the proof. ■

The previous lemma is useful to prove the following result.

Proof of Proposition 12:

Proof. Suppose that the $\mathcal{OA}$ attains the unstable matching μ as a SPE outcome. Assume that μ is not individually rational: If $(u, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$ the student s_k has a profitable deviation with $m'(s_k) = u$. We have a similar case when $(\mu(s_k), u)\, P_c\, (\mu(s_k), \mu(s_l))$. Consider the third possibility $(u, u)\, P_c\, (\mu(s_k), \mu(s_l))$. We know that $(u, u) \neq (\mu(s_l), \mu(s_k))$, hence by Lemma 1 it is satisfied either $u \succ_{s_k} \mu(s_k)$ or $u \succ_{s_l} \mu(s_l)$. Assume w.l.g. that $u \succ_{s_k} \mu(s_k)$, by responsiveness $u \succ_{s_k} \mu(s_k)$ implies $(u, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$, hence $m'(s_k) = u$ is a profitable deviation for s_k. A contradiction, hence the matching μ has to be individually rational.

The previous argument implies that by assumption there has to exist at least one blocking coalition, say $\{c = (s_k, s_l), (h_p, h_q)\}$. There are three possible cases:

1. Assume either $h_p = \mu(s_k)$ or $h_q = \mu(s_l)$. Consider w.l.g. that $h_q = \mu(s_l)$ then $m'(s_k) = h_p$ is a profitable deviation for s_k. Since no other agent deviates if $h_p \in H$, then $M'(h_p) = M(h_p) \cup \{s_k\}$ and $\mu(h_p) \in M(h_p)$. So, the hospital h_p will choose the candidate $C_{h_p}(M'(h_p)) = s_k$ which confirms that $m'(s_k) = h_p$ is a profitable deviation for s_k. A contradiction.

2. Assume that $h_p \neq \mu(s_k)$, $h_q \neq \mu(s_l)$ and $(h_p, h_q) \neq (\mu(s_l), \mu(s_k))$. By Lemma 1 (case 3), we know that $(h_p, h_q)\, P_c\, (\mu(s_k), \mu(s_l))$ implies either $h_p \succ_{s_k} \mu(s_k)$ or $h_q \succ_{s_l} \mu(s_l)$. Suppose that $h_p \succ_{s_k} \mu(s_k)$, so by responsiveness $(h_p, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$. Hence, as in the previous case the message $m'(s_k) = h_p$ is a profitable deviation for s_k. A contradiction.

3. Assume that $(h_p, h_q) \neq (\mu(s_l), \mu(s_k))$ and either $h_p = \mu(s_l)$ or $h_q = \mu(s_k)$. By Lemma 1 (case 1), $(h_p, h_q)\, P_c\, (\mu(s_k), \mu(s_l))$ implies either $h_p \succ_{s_k} \mu(s_k)$ or $h_q \succ_{s_l} \mu(s_l)$. So, by responsiveness it is possible either $(h_p, \mu(s_l))\, P_c\, (\mu(s_k), \mu(s_l))$ or $(\mu(s_k), h_q)\, P_c\, (\mu(s_k), \mu(s_l))$. As in previous cases, any of the students s_k or s_l has a profitable deviation. A contradiction.

There is only one more possibility, the blocking coalition: $\{c = (s_k, s_l), (\mu(s_l), \mu(s_k))\}$. However, we have already shown in Example 1 that it is easy to construct a profile of strategies that supports this kind of instability in SPE. This completes the proof. ■

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